UKMT

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

G. H. Hardy

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

G. H. Hardy

Sheet 1

Solutions and comments

This programme of the Mentoring Scheme is named after G. H. Hardy (1877–1947).

See http://www-history.mcs.st-andrews.ac.uk/Biographies/Hardy.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

*Mentoring Scheme, UK Mathematics Trust, School of Mathematics,
University of Leeds, Leeds LS2 9JT*

☎ 0113 343 2339   mentoring@ukmt.org.uk   www.ukmt.org.uk

Version 2.0, Oct 2020

1. Set the scene for your work on this scheme by discovering some facts about the life and work of the twentieth century English mathematician G H Hardy.

Solution

See the cover sheet for a source of information.

2. How many times between 6 am and 6 pm do the minute hand and hour hand of a clock lie one above the other?

Solution

There are many ways to answer this question and a straightforward method is to work out where each hand is at a particular time. However, it is slightly easier to calculate the *relative* angular speed of the two hands.

The minute hand makes one revolution per hour.

The hour hand makes $\frac{1}{12}$ of a revolution each hour.

The two hands are moving in the same direction, so their relative angular speed is the difference of these two values, that is $\frac{11}{12}$ of a revolution each hour.

Thus in any twelve-hour period, the angle between the hands goes through 11 cycles. In particular, they overlap 11 times.

3. Does there exist a square number whose digit sum is 123? Justify your answer.

Solution

For an integer to be a square number, every prime in its prime factorisation must appear to an even power. The reference to the digit sum suggests the tests for divisibility by 3 and by 9. Suppose the positive integer N has digit sum $123 = 3 \times 41$. Since the digit sum is divisible by 3, N itself must be divisible by 3. However, 9 does not divide 123 and therefore N is not divisible by 9. Hence N is not divisible by any higher power of 3. (If this were true, then what is the contradiction that would follow?) We conclude that the prime factorisation of N includes a single factor of 3 only and consequently N cannot be a square number.

4. The collection $\{8, 9, 12\}$ is a set of three positive integers with the property that, given any two of these integers, their greatest common divisor (gcd) is equal to their difference:

- $\gcd(8, 9) = 1 = 9 - 8$;
- $\gcd(8, 12) = 4 = 12 - 8$;
- $\gcd(9, 12) = 3 = 12 - 9$.

Can you find a set of four integers with the same property? What about five? Or six?

Solution

It is actually possible to add a fourth number to the set $\{8, 9, 12\}$ and still have the property that

the gcd of any pair is equal to their difference (choose __ or __). But there is no hope that we can go on doing this indefinitely. This is because, if our set contains the number n, then it cannot also contain any number larger than $2n$ (why?). So some other idea is needed.

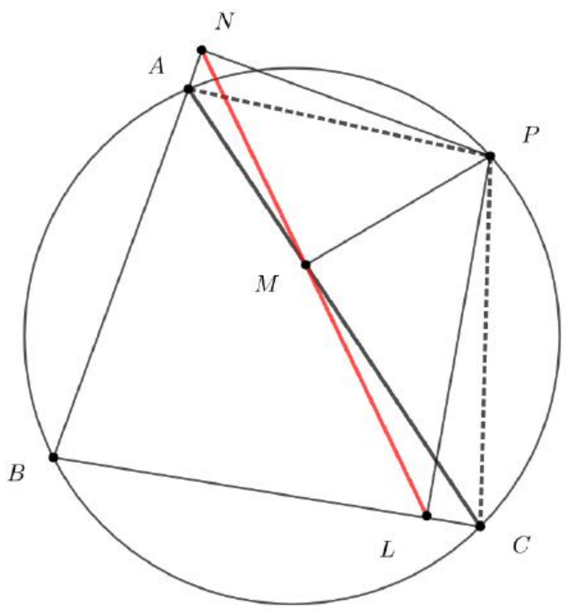
Suppose that we have two numbers a and b, whose greatest common divisor is equal to their difference d. Consider the numbers $a + k$ and $b + k$, where k is a multiple of d. The difference between $a + k$ and $b + k$ is still d. Also $a + k$ and $b + k$ are both divisible by d, because a, b and k are all divisible by d. And if h is any number that divides both $a + k$ and $b + k$, then it must also divide their difference d. This is what is required for d to be the *greatest* common divisor of $a + k$ and $b + k$.

This means that if the set $\{a_1, a_2, \ldots, a_n\}$ has the required property, then so does the set (with one more element) $\{P, a_1 + P, a_2 + P, \ldots, a_n + P\}$, where P is the product of all the a_i. By repeating this construction, we can produce a set, with the desired property, that is as large as we like.

5. The point P lies on the circumcircle of the triangle ABC. Perpendiculars PL, PM and PN are drawn to the sides BC, CA and AB respectively. If necessary, the sides of the triangle are produced (extended). Prove that the points L, M and N are collinear (lie on the same straight line).

Solution

A diagram of the configuration follows.



What is the overall strategy to be? The two parts of the desired straight line meet at M, so a simple approach is to focus on that point and to use the converse of the elementary theorem that the angles on a straight line add to $180°$. (Fortunately this converse is also a valid result.)

When you draw your own diagram, you may like to try out several versions, depending on the nature of the triangle ABC and the position of P relative to it. Note that, in the figure above, the chords PA and PC have been added. Since we aim to prove that LMN is a straight line, care

must be taken that the result is not assumed during the proof, even though it looks plausible in the diagram.

Clearly the quadrilateral $ABCP$ is cyclic but are there any other cyclic quadrilaterals in the figure? The quadrilateral $AMPN$ has the property that $\measuredangle AMP = \measuredangle PNA = 90°$. Thus $AMPN$ is cyclic because it has a pair of opposite angles that add to $180°$. Note that we have used the *converse* of "opposite angles of a cyclic quadrilateral add to 180°", also a valid result. Again in $PMLC$, $\measuredangle PMC = \measuredangle PLC = 90°$. Hence by the converse of "angles in the same segment", $PMLC$ is a cyclic quadrilateral.

Next observe that $\measuredangle NAP$ is an external angle of the cyclic quadrilateral $ABCP$. This is equal to the opposite internal angle so $\measuredangle NAP = \measuredangle BCP = \measuredangle LCP$. (This useful result can be justified by using angles on a straight line at A, followed by the opposite angles property of a cyclic quadrilateral.) In the right-angled triangle LCP, we have $\measuredangle LCP = 90° - \measuredangle CPL$. Combining this with the previous result, $\measuredangle NAP = 90° - \measuredangle CPL$. However, $PMLC$ is cyclic so $\measuredangle CPL = \measuredangle CML$ by angles in the same segment and we conclude that $\measuredangle NAP = 90° - \measuredangle CML$.

Considering the cyclic quadrilateral $AMPN$, we see that $\measuredangle NMP = \measuredangle NAP$, again by angles in the same segment. We now have two angles equal to $\measuredangle NAP$. Equating them yields $90° - \measuredangle CML = \measuredangle NMP$ or $\measuredangle CML + \measuredangle NMP = 90°$.

Finally, use this last result to find the following sum of angles at M: $\measuredangle CML + \measuredangle PMC + \measuredangle NMP$. Because $\measuredangle PMC = 90°$, the required sum is $90° + 90° = 180°$. This shows that the points L, M and N are indeed collinear.

6. Isaac attempts six questions on an Olympiad paper in order. Each question is marked on a scale of 0 to 10. He never scores more in a later question than in any earlier question. How many possible sequences of six marks can he achieve?

[British Mathematical Olympiad Round 1 2009 Question 3]

SOLUTION

This question would be much simpler if all Isaac's scores were different, so we have to deal with the fact that there may be scores that are equal. One way to do this would be to graph Isaac's scores on a bar chart, with possible marks on the vertical axis and the six questions along the horizontal axis. The bar chart will look like an irregular descending staircase. Walking down it involves a sequence of fifteen steps, of which exactly six are to the right and nine are down. It is then possible to calculate the number of such staircases.

A different solution is presented in full because it demonstrates a trick that is often useful. Consider what happens when we add 6 to Isaac's score for Q1, 5 to his score for Q2, 4 to his score for Q3, 3 to his score for Q4, 2 to his score for Q5 and 1 to his score for Q6. We now have a sequence of six scores between 16 (the maximum possible for Q1) and 1 (the minimum possible for Q6). Moreover all these new scores are different because they are strictly decreasing in order from Q1 to Q6. Furthermore, by doing the appropriate subtractions, it is possible to recover the original scores.

The new scores form a subset of the integers from 1 to 16. The number of such subsets is the number of ways of choosing 6 integers from these 16 integers. This is provided by the binomial

coefficient $\binom{16}{6}$, which can be computed as follows.

$$\binom{16}{6} = \frac{16 \times 15 \times 14 \times 13 \times 12 \times 11}{6 \times 5 \times 4 \times 3 \times 2 \times 1} = \frac{16 \times 14 \times 13 \times 11}{4} = 4 \times 14 \times 13 \times 11 = 4 \times 182 \times 11.$$

Continuing the calculation, we have $728 \times 11 = 7280 + 728 = 8008$.
Note: The full details are given to show how there is no need to stop an argument to carry out long multiplications. Judicious cancelling helps too.

7. Let n be a positive integer. Mark has a deck of n cards numbered 1 to n. He aims to place the cards one at a time on a table such that, at each step, the mean of the numbers on the cards on the table is an integer.

For which n can Mark complete the task?

Solution

Let us do some small examples to try and see what is going on.

- When $n = 1$, Mark can just place 1 down on the table.
- When $n = 2$, Mark can place either 1 or 2 on the table first but then upon placing the other one the mean of the numbers on the table is $\frac{1+2}{2} = \frac{3}{2}$ is not an integer.
- When $n = 3$, Mark can place 1, then 3, then 2 on the table (the corresponding averages being 1, 2 and 2). Alternatively he could place 3 then 1 then 2 (the corresponding averages being 3, 2 and 2).
- When $n = 4$, Mark runs into the same problem as for $n = 2$: once he has placed all the cards on the table their average will be $\frac{1+2+3+4}{4} = \frac{5}{2}$ which is not an integer.

So Mark can complete the task for $n = 1, 3$ but not $n = 2, 4$; this suggests that we should investigate what the average is once all the cards have been placed on the table.

The sum of the first n integers is $\frac{n(n+1)}{2}$, so once all the cards are placed on the table the average will be $\frac{n+1}{2}$. When n is even this is not an integer so the task is impossible for Mark in this case.

We are left to consider what happens when n is odd and at least 5. At this point, either we want to give a general method that allows Mark to complete the task for lots of odd numbers or give a general reason for why Mark cannot complete the task for lots of odd numbers. It would help to try another example to see which we should aim for!

When $n = 5$, Mark can place three of the five cards on the table (there are quite a few ways e.g. 1, then 3, then 2 or 3, then 5, then 1) but the fourth card always seems to thwart him: it seems to be impossible to get the sum (of the first four cards) to be a multiple of 4.

Let us try and understand what is going on here. Suppose Mark can complete the task: once all five cards are placed the sum is 15. After the first four cards are placed the sum is a multiple of 4 and between 10 and 14 so must be 12. So the last card placed by Mark must be a 3. What about the sum of the first three cards placed? Well the sum is a multiple of 3 and between 7 and 11 so must be 9. So the penultimate card placed by Mark must also be a 3 which is impossible.

Now we attempt to generalise this argument for all odd n at least 5: we can write $n = 2k - 1$ where k is an integer that is at least 3. Once all n cards are placed the sum is equal to $\frac{n(n+1)}{2} = k(2k - 1)$.

The sum of the first $n-1$ cards placed is between $k(2k-1)-(2k-1) = 2k^2-3k+1$ and $k(2k-1)-1 = 2k^2-k-1$ and is a multiple of $n-1 = 2k-2$. What multiples of $2k-2$ lie in the range we want? Well $k(2k-2) = 2k^2-2k$ certainly does, but how about the one before and after? $(k-1)(2k-2) = 2k^2-4k+2$ is too small (check this!), while $(k+1)(2k-2) = 2k^2-2$ is too big (check this!). Thus the sum of the first $n-1$ cards placed must be $k(2k-2)$ and so the last number Mark placed has to be $k(2k-1)-k(2k-2) = k$.

The sum of the first $n-2$ cards placed is between $k(2k-2)-(2k-1) = 2k^2-4k+1$ and $k(2k-2)-1 = 2k^2-2k-1$ and is a multiple of $n-2 = 2k-3$. $k(2k-3) = 2k^2-3k$ is a multiple of $n-2$ lying in the range. $(k-1)(2k-3) = 2k^2-5k+3$ is too small (check this!) and $(k+1)(2k-3) = 2k^2-k-3$ is too big (check this!). Hence the sum of the first $n-2$ cards placed must be $k(2k-3)$ and so the penultimate number Mark placed has to be $k(2k-2)-k(2k-3) = k$ as well which is impossible.

In conclusion, Mark can complete the task only if n is 1 or 3.

8. Find the last non-zero digit of 50!

Solution

Note: we present two solutions here, the first of which was devised by a mentee. This has been edited a little to make the simple calculations even simpler. The second could be termed the classic solution. A third solution, a rather plodding straightforward approach, is mentioned .

First solution

First note that the power of 5 in the prime factorisation of 50! is 12 and that the power to which 2 appears is greater than 12. Thus the required final digit is given by $\dfrac{50!}{10^{12}} \pmod{10}$. We can write

$$\begin{aligned} & 50! \\ = \; & (1\times2\times3\times4)\times5\times(6\times7\times8\times9)\times10\times\cdots\times(46\times47\times48\times49)\times50 \\ = \; & (1\times2\times3\times4)\times(6\times7\times8\times9)\times\cdots\times(46\times47\times48\times49)\times(5\times10\times15\times\cdots\times50). \end{aligned}$$

To see the effect of dividing by 10^{12}, observe first that

$$(1\times2\times3\times4)\times(6\times7\times8\times9) = 2^7\times3^4\times7.$$

It follows that

$$5\times10\times15\times\cdots\times50 = 5^{12}(1\times2\times3\times4)\times(6\times7\times8\times9)\times2 = 5^{12}\times2^8\times3^4\times7.$$

Thus, when 50! is divided by 10^{12}, the first two four-term brackets and the last bracket containing the multiples of 5 mop up 12 powers of 10, leaving

$$\begin{aligned} & 50!/10^{12} \\ = \; & 2^3\times(3^4\times7)^2\times(11\times12\times13\times14)\times\cdots\times(46\times47\times48\times49). \end{aligned}$$

Recall that we need to find the remainder of this product modulo 10. Each of the last four pairs of brackets is congruent to $(1\times2\times3\times4)\times(6\times7\times8\times9) \pmod{10}$. We already know that

$$(1\times2\times3\times4)\times(6\times7\times8\times9) = (2^7\times3^4\times7) = 128\times81\times7 \equiv 8\times7 \equiv 6 \pmod{10}.$$

Since any power of an integer with last digit 6 itself has last digit 6, the product of these four pairs is congruent to 6 modulo 10. Combining these findings yields the overall result:

$$50!/10^{12} \equiv 2^3 \times (3^4 \times 7)^2 \times 6 (\text{mod } 10).$$

Continuing to work modulo 10 gives

$$50!/10^{12} \equiv 8 \times (81 \times 7)^2 \times 6 \equiv 8 \times 7^2 \times 6 \equiv 8 \times 9 \times 6 \equiv 72 \times 6 \equiv 2 (\text{mod } 10).$$

We conclude that the last non-zero digit of 50! is 2. (In these final calculations, note the benefit conferred by 3^4 having 1 as its last digit.)

Second, classic, solution

We first find the prime factorization of 50! To do this we begin by finding the highest power of 2 that is a factor of the product

$$1 \times 2 \times 3 \times 4 \times 5 \times \cdots \times 50 \qquad (1)$$

that defines 50! and then generalize the method.

It is convenient to use the notation $\theta(p)$ for the highest power of the prime p that divides 50!

Every second term in the product (1) has a factor of 2, and so contributes 1 to $\theta(2)$. There are $\dfrac{50}{2} = 25$ of these terms.

Every fourth term in the product (1) has a factor of 2^2, and so, in addition to the factor 2 that we have already counted, contributes a further 1 to $\theta(2)$. There are $\left\lfloor \dfrac{50}{2^2} \right\rfloor = 12$ of these terms. [Note that here $\lfloor x \rfloor$ is the *integer part* of x, that is, the largest integer that does not exceed x. For example $\left\lfloor \dfrac{50}{2^2} \right\rfloor = \lfloor 12.5 \rfloor = 12$].

Similarly there are $\left\lfloor \dfrac{50}{2^3} \right\rfloor$ terms in the product which have a factor 2^3, and which each therefore contributes an additional 1 to $\theta(2)$, and so on.

We therefore see that

$$\theta(2) = \left\lfloor \frac{50}{2} \right\rfloor + \left\lfloor \frac{50}{2^2} \right\rfloor + \left\lfloor \frac{50}{2^3} \right\rfloor + \left\lfloor \frac{50}{2^4} \right\rfloor + \left\lfloor \frac{50}{2^5} \right\rfloor.$$

This sum stops after the fifth term because for $k > 5$, $\dfrac{50}{2^k} < 1$ and therefore $\left\lfloor \dfrac{50}{2^k} \right\rfloor = 0$. Hence

$$\theta(2) = 25 + 12 + 6 + 3 + 1 = 47.$$

Similarly, in general, the highest power of the prime p that divides 50! is given by

$$\theta(p) = \left\lfloor \frac{50}{p} \right\rfloor + \left\lfloor \frac{50}{p^2} \right\rfloor + \left\lfloor \frac{50}{p^3} \right\rfloor + \ldots + \left\lfloor \frac{50}{p^k} \right\rfloor, \qquad (2)$$

with the sum ending after k terms, where k is the largest integer such that $p^k \leqslant 50$. [Note that, if we replace 50 by n in this formula, then we obtain the formula for the highest power of the prime p that divides n!]

Since the largest prime which does not exceed 50 is 47, it follows that

$$50! = 2^{\theta(2)} \times 3^{\theta(3)} \times 5^{\theta(5)} \times \cdots \times 47^{\theta(47)}.$$

We leave it to the reader to use (2) to work out the values of the terms $\theta(p)$ that occur in this factorization and thus deduce that

$$50! = 2^{47} \times 3^{22} \times 5^{12} \times 7^8 \times 11^4 \times 13^3 \times 17^2 \times 19^2 \times 23^2 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47.$$

Because the highest power of 5 that divides 50! is 5^{12}, we see that 10^{12} is the highest power of 10 that divides 50! and that, as $10^{12} = 2^{12} \times 5^{12}$, we can deduce that

$$\frac{50!}{10^{12}} = 2^{35} \times 3^{22} \times 7^8 \times 11^4 \times 13^3 \times 17^2 \times 19^2 \times 23^2 \times 29 \times 31 \times 37 \times 41 \times 43 \times 47. \qquad (3)$$

The final non-zero digit of 50! is the units digit of the number on both sides of equation (3). We can now use the language of modular arithmetic and say that the digit we seek is the value of this number (mod 10). The value of a positive integer (mod 10), is the value of its units digit. Thus from (3) we have that

$$\frac{50!}{10^{12}} \equiv 2^{35} \times 3^{22} \times 7^8 \times 1^4 \times 3^3 \times 7^2 \times 9^2 \times 3^2 \times 9 \times 1 \times 7 \times 1 \times 3 \times 7 (\text{mod } 10). \qquad (4)$$

We can now drop the factors that are equal to 1, and replace 9 by 3^2, to rewrite equation (4) as

$$\frac{50!}{10^{12}} \equiv 2^{35} \times 3^{22} \times 7^8 \times 3^3 \times 7^2 \times (3^2)^2 \times 3^2 \times 3^2 \times 7 \times 3 \times 7 (\text{mod } 10). \qquad (5)$$

Hence, collecting powers of the primes 2, 3 and 7, we have

$$\frac{50!}{10^{12}} \equiv 2^{35} \times 3^{34} \times 7^{12} (\text{mod} 10). \qquad (6)$$

We now note that (mod 10) the powers of 2 are $2^1 \equiv 2$, $2^2 \equiv 4$, $2^3 \equiv 8$, $2^4 \equiv 6$, $2^5 \equiv 2$ and so on, thus forming a cycle of length 4. Similarly the powers of 3 (mod 10) are 3, 9, 7, 1, 3 ,.. and the powers of 7 (mod 10) are 7, 9, 3, 1, 7, ..., in each case forming a cycle of length 4.

Thus, in equation (6) we can replace the exponents by their remainders when divided by 4. Therefore, it follows from (6) that

$$\frac{50!}{10^{12}} \equiv 2^3 \times 3^2 \times 7^0 \equiv 8 \times 9 \times 1 \equiv 72 \equiv 2 (\text{mod} 10).$$

We deduce that the final non-zero digit of 50! is 2.

Alternative approach
Like the first solution, this breaks up the factors of 50! into groups, each of the form

$$(10k + 1)(10k + 2)(10k + 3)(10k + 4)(10k + 5)(10k + 6)(10k + 7)(10k + 8)(10k + 9)(10k + 10),$$

where $k = 0, 1, 2, 3, 4$. Then the method just works away at this algebra, throwing out powers of 10 wherever possible. With enough care it is possible to avoid carrying out most of the algebraic manipulation.

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

G. H. Hardy

Sheet 2

Solutions and comments

This programme of the Mentoring Scheme is named after G. H. Hardy (1877–1947).
See http://www-history.mcs.st-andrews.ac.uk/Biographies/Hardy.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

Mentoring Scheme, UK Mathematics Trust, School of Mathematics,
University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 0113 343 5500 mentoring@ukmt.org.uk www.ukmt.org.uk

1. One hundred students, all of different heights, are arranged in a square of 10 rows by 10 columns. In each row the tallest student is selected, and the shortest of these tall students is labelled A. In each column the shortest student is selected, and the tallest of these short students is labelled B. If A and B are different people, which of them is taller and why?

Solution

Label the student in row i $(1 \leqslant i \leqslant 10)$ and row j $(1 \leqslant j \leqslant 10)$ by d_{ij}. Suppose the student labelled A is in row k and the student labelled B is in column l but not in row k. Consider the student d_{kl} who is in row k and column l. Since A is the tallest student in row k, A is taller than d_{kl}. Again, since B is the shortest student in column l, B is shorter than d_{kl}. Since the students are of different heights, this implies that A is taller than B.

This argument will work unless both A and B are in the same row or the same column. Suppose first that they are both in row k. Student A is the tallest student in this row, so is taller than B. On the other hand, if A and B are both in column l, then since B is the shortest student in this column, A is still taller than B. Now all cases have been dealt with and in each A was taller than B.

2. *a*) Show that every perfect square leaves remainder either 0 or 1 when we divide by 4.

b) The Swiss mathematician Leonhard Euler was born in 1707. Can you find integers a, b such that $a^2 + b^2 = 1707$?

c) The French mathematician Sophie Germain was born in 1776. Can you find integers a, b with the property that $a^2 + b^2 = 1776$?

Solution

a) Any positive integer can be expressed in the form $4k + r$, where $k \geqslant 0$ and r are integers and $0 \leqslant r \leqslant 3$.

- When $r = 0$, $(4k + 0)^2 = 4(4k^2) + 0$.
- When $r = 1$, $(4k + 1)^2 = 4(4k^2 + 2k) + 1$.
- When $r = 2$, $(4k + 2)^2 = 4(4k^2 + 4k + 1) + 0$.
- When $r = 3$, $(4k + 3)^2 = 4(4k^2 + 6k + 2) + 1$.

These four cases show that the possible remainders are 0 and 1.

b) Since 1707 is not a large number, we might test all the squares a^2 up to 1707 to see whether $1707 - a^2$ is a square number. (It would, of course, be best to do this via a spreadsheet or computer program!) However, even though such a procedure would show that there are no integer solutions to the equation $a^2 + b^2 = 1707$, it does not really give any insight into *why* there are no solutions. Happily the previous part of the question will help.

Since every square is congruent to 0 or 1 modulo 4, we see that if a and b are integers then $a^2 + b^2$ is congruent to __ or __ or __ modulo 4 but definitely not to 3 modulo 4. However $1707 \equiv 3 \pmod 4$ so there are no solutions for a and b.

c) This time we see that $1776 \equiv 0 \pmod 4$. It is certainly possible for a multiple of 4 to be a sum of two squares but that does not mean that every multiple of 4 is a sum of of two squares. In fact it will turn out that this one is not.

How can we make a multiple of 4 by adding two squares? First the integers that we square have to be even. (What would happen if at least one of them were odd?) Let $a = 2k$ and $b = 2l$ for some integers k and l. Then $a^2 + b^2 = 1776$ becomes $4k^2 + 4l^2 = 1776$, that is $k^2 + l^2 = 444$. Now 444 is is again a

multiple of 4 so the argument can be repeated. Setting $k = 2m$ and $l = 2n$ yields $4m^2 + 4n^2 = 444$ or $m^2 + n^2 = 111$.

One way to show that this last equation has no solutions is to work modulo 8 instead of modulo 4. An argument similar to that in part *a)* shows that every square is congruent to ___ or ___ or ___ modulo 8. But $111 \equiv 7 \pmod 8$ and, on checking the various cases for $m^2 + n^2$, we see that $m^2 + n^2 = 111$ has no solutions. Hence the same is true of $a^2 + b^2 = 1776$.

Remark: In questions like this it is often worth experimenting with different moduli to see which would give the most efficient solution. You may want to experiment yourself or discuss this aspect with your mentor.

3. Six points are arranged around a circle and any two of them define a chord. Four of the chords so defined are picked at random. What is the probability that these four chords form a convex* quadrilateral?
*A **convex** polygon is one whose diagonals all lie inside its perimeter.*

Solution

The diagrams below illustrate two cases that are certainly not convex quadrilaterals. The original six points are numbered in order around the circumference of the circles.

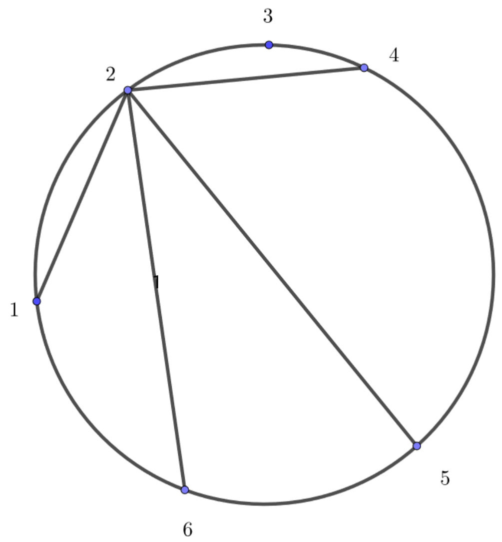


Diagram (i)

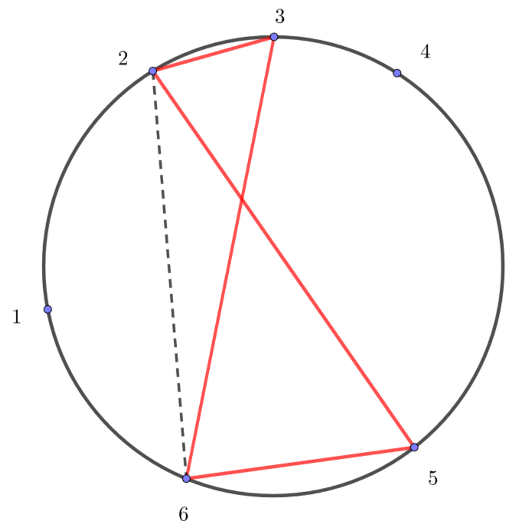


Diagram (ii)

A couple of fairly unusual cases are illustrated above. In Diagram (i) the four chords do not form a quadrilateral. In Diagram (ii) the red chords form a quadrilateral but it is not convex: for example the diagonal 26 lies outside the quadrilateral.

First we need to know how many chords there are. Since each is defined by choosing two out of the six given points,

$$\text{total number of chords is } \binom{6}{2} = \frac{6 \times 5}{2 \times 1} = 15.$$

Next we have to choose 4 chords from these 15:

$$\text{number of ways of choosing 4 chords is } \binom{15}{4} = \frac{15 \times 14 \times 13 \times 12}{4 \times 3 \times 2 \times 1} = 15 \times 7 \times 13 = 15 \times 91.$$

Next observe that the only way to obtain a convex quadrilateral is to select 4 points from the original 6 and to join these 4 in order around the circle. (This produces a more usual type of diagram with a quadrilateral

inscribed in a circle.) Furthermore there is a one-one correspondence between each set of 4 points and the convex quadrilateral formed from them. Hence the number of convex quadrilaterals is the number of ways of choosing 4 points out of 6. This is

$$\binom{6}{4} = \binom{6}{2} = 15.$$

Hence

the probability that the four chords form a convex quadrilateral is $\dfrac{15}{15 \times 91} = \dfrac{1}{91}$.

4. *Before starting this question, you may want to look back at Question 8 of Hanna Neumann Sheet 4. Here the concepts of* ***graph*** *and* ***complete graph*** *were introduced. A* ***Hamiltonian cycle*** *in a graph is a route visiting every vertex exactly once before returning to the starting vertex.* In what follows we shall say that two Hamiltonian cycles in the same graph are ***non-overlapping*** if they have no edge in common.

a) Explain why the complete graph with n vertices cannot contain two non-overlapping Hamiltonian cycles if $n < 5$.

b) For each of the cases $n = 5$ and $n = 6$ find two non-overlapping Hamiltonian cycles in the complete graph with n vertices.

c) For $n \geqslant 5$ prove that it is always possible to find two non-overlapping Hamiltonian cycles in the complete graph with n vertices.

Solution

a) In a complete graph every vertex shares an edge with every other vertex, every edge connects two vertices, so the total number of edges in a complete graph with n vertices is N where

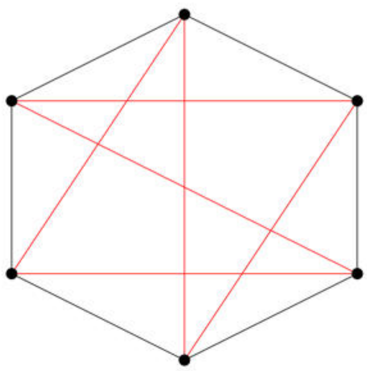
$$N = \frac{n(n-1)}{2}.$$

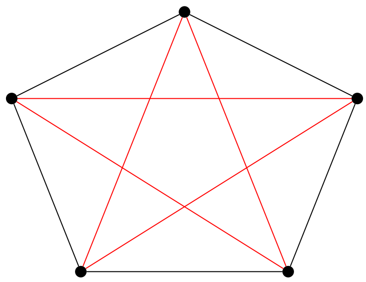
Now a Hamiltonian cycle in the graph arrives at, and then leaves, each of the n vertices. It therefore contains $2n$ edges. For a Hamiltonian cycle to exist we must have

$$N = \frac{n(n-1)}{2} \geqslant 2n.$$

This is equivalent to $n(n-5) \geqslant 0$. In other words, for a Hamiltonian cycle to exist we must have $n \geqslant 5$.

b) The diagrams below below show a possible solution in each case.





c) ***First solution:*** For $n \geq 5$, without loss of generality, we can number the vertices and place then in that very order at the vertices of an n-gon. So the perimeter of this n-gon serves as our first Hamiltonian. For n's with a coprime integer j, say, satisfying $2 \leqslant j \leqslant n-2$, we could go round the vertices in jumps of j and trace a path which we know will visit all vertices before it returns to the vertex of origin. *(See Remark 1 below.)* So this is our second Hamiltonian cycle, which does not overlap the first because it uses only edges of the graph that are inside the n-gon.

Note that jumps of length $n-1$ are ruled out. (Why?) This immediately causes a problem in the case $n = 6$. There are just two integers smaller than 6 which are coprime to it, namely 1 and 5. Jumps of 1 would reproduce the first Hamiltonian, while 5 is excluded because $5 = 6 - 1$. In part *b)* the first diagram shows that a (red) non-overlapping Hamiltonian cycle can nevertheless be found. In the second diagram in part *b)* the standard method described above is applied to a pentagon. Now we explore other odd integers: do they behave in the same way?

Initial case: $n \geqslant 5$ is odd. Jumps of $n-1$ have been ruled out, so try jumps of $n-2$. Is it the case that n and $n-2$ are coprime? Suppose there were a prime divisor p of both integers. Since n is odd, $n-2$ is also odd and so too is any common prime divisor. (Make sure you understand why this is.) However, if p divides both n and $n-2$, then p divides $n-(n-2) = 2$. Contradiction: this is impossible for an odd prime. Thus we conclude that n and $n-2$ are coprime and $n \geqslant 5$ implies $n-2 > 2$. (Note that, in the first diagram above, the jumps can be regarded as having length $5-2=3$.)

Now we turn to the cases where n is even. It is often useful to consider separately integers of the form $n = 2^k$ and $n = 2^k m$, where m is an odd integer. Here the odd integer 3 may provide an additional special case, suggested by the anomalous behaviour of $n = 6$. In examining these cases, we shall try to follow the sort of method used above for odd n.

- **Case (i)**: $n = 2^k$, where $k \geqslant 3$. Since n and $n-2$ share the prime factor 2, try $n-3$. The only prime dividing 2^k is 2 and 2 does not divide $2^k - 3$. Hence n and $n-3$ have no prime factor in common and $k \geqslant 3$ implies that $n-3 > 2$.

- **Case (ii)**: $n = 2^k m$, where $k \geqslant 1$ and $m \geqslant 5$ is odd. The basic choice of $n-2$ is ruled out because of the power of 2 so try $m-2$. An argument similar to that used in the initial case shows that $m-2$ works and it is left to the reader.

- **Case (iii)**: $n = 2^k \times 3$, where $k \geqslant 2$. The only prime factors of n are 2 and 3, so $n-2$ and $n-3$ are ruled out. Considering some small values of k may suggest a way forward. The sequence of values of $2^k \times 3$ starts 12, 24, 48, 96, ... First concentrate on avoiding factors of 2, so strip out the factors of 3. The sequence of powers of 2 starts 4, 8, 16, 32, ... Since these produce odd numbers if we subtract 1, perhaps the sequence $2^k - 1$ might produce integers coprime to n. This sequence starts 3, 7, 15, 31, ... Parts of this look promising but two out of these four terms are divisible by 3. So there are cases where 3 divides both n and $2^k - 1$. However, if $2^k - 1$ is a multiple of 3,

then $2^k + 1$ certainly is not. Thus a suitable coprime integer can always be found. As for the inequalities: since $k \geqslant 2$, $2^k - 1 > 2$. We also need to check that $2^k + 1 \leqslant 2^k \times 3 - 2$. But this is equivalent to $3 \leqslant 2^k(3-1) = 2^{k+1}$, which is certainly true for $k \geqslant 2$.

Remark 1: This statement about a coprime integer can appear obvious. But you may like to have a go at writing down ***exactly*** *why this is the condition we need. Alternatively, you could discuss it with your mentor.*
Remark 2: You may well have thought that the discussion of the various cases became very bitty. You may also have wondered if there were a result that would polish them all off together. There is indeed such a result and it involves ***Euler's totient function.*** *This function tells us how many integers smaller than a given n are coprime to n. However, for those who have not met it before, its justification requires some sophisticated mathematics and that is why the simpler arguments were presented.*

Second solution (suggested by a mentee): This solution is based on the principle of induction. When $n = 5$ we know there are 2 non overlapping Hamiltonian cycles, the outer pentagon and the inner star. Now we assume that there are 2 non overlapping Hamiltonian cycles for a complete graph of n vertices. We can add a vertex to this graph, by randomly choosing an edge on one of the cycles to be eliminated in order to insert the new vertex and make it adjacent to these 2 vertices originally incident to the removed edge. These 2 vertices must be incident to 2 pairs of distinct edges which belong to the other Hamiltonian cycle. We know this because the 2 Hamiltonian cycles are non overlapping. For $n \geq 5$, there is at least 1 edge in the second Hamiltonian cycle which is non adjacent to the already removed edge. This tells us that there is at least one edge in the second Hamiltonian cycle which can be removed in order to include the new vertex in its cycle without having any overlapping with the augmented first cycle.
So what we have shown here is for $n \geq 5$, if there are 2 non overlapping Hamiltonian cycles for a complete graph of degree n, it must also be true for a complete graph of degree $n + 1$. So it is true for all $n \geq 5$. This complete our proof by induction.

5. The most common type of football is a semi-regular polyhedron. This is a polyhedron whose faces consist of more than one type of regular polygon and where the arrangement of faces around every vertex is of same type. The polygons in a football are pentagons and hexagons. Determine how many pentagons and how many hexagons are needed. You may find the following information useful:

- The sum of all the internal angles of the polygons meeting at a vertex must be less than 360°. Here think of cutting along an edge incident to a vertex and flattening out the polygons around that vertex.
- Euler's formula $F - E + V = 2$ for a polyhedron. Here F is the number of faces, E is the number of edges (where two sides of two distinct polygons meet) and V the number of vertices.

Remark: Of course, a real football is not an actual polyhedron because of the internal pressure.

Solution

Two faces meet in an edge. A vertex must be a meeting of at least three faces. If four faces, which are pentagons or hexagons, were to meet at a vertex, the sum of the internal angles would be greater than or equal to $432° > 360°$. So we know that the semi-regular polyhedron we are after has three faces meeting at a vertex. Also these three faces must be be either two pentagons and one hexagon, or two hexagons and one pentagon. (What would happen if you tried to fit three hexagons around a vertex? What would happen if you tried to fit three pentagons around a vertex?)

To pursue the former case, we start with a pentagon: each of its sides must be adjacent to a side of a

pentagon and a hexagon in an alternating fashion. As a pentagon has an odd number of sides, the fifth neighbour would also be adjacent to the first neighbour. Contradiction: so this case can be dropped.

We have no immediate reason to reject the possibility of two hexagons and one pentagon at a vertex. So we proceed with Euler's formula. Let p denote the number of faces which are pentagons and h the number of faces that are hexagons. First it is obvious that $F = p + h$.

Then $2E = 5p + 6h$. To prove this, it suffices to count the total number of sides of all the polygonal faces in two different ways. Since there are p pentagons and h hexagons, the total number of sides is $5p + 6h$. Again, since an edge corresponds to two sides meeting, the total number of sides is $2E$.

Finally, we have $2E = 3V$. To prove this, it suffices to count all the end points provided by all the edges. Since each edge has two ends, this number is $2E$. However, it is also the case that three edges meet at each vertex. Thus the total numbers of end points is also $3V$.

Putting everything in terms of p and h into Euler's formula:

$$p + h - \frac{5p + 6h}{2} + \frac{5p + 6h}{3} = 2.$$

In this equation the h's cancel out and leave $p = 12$.

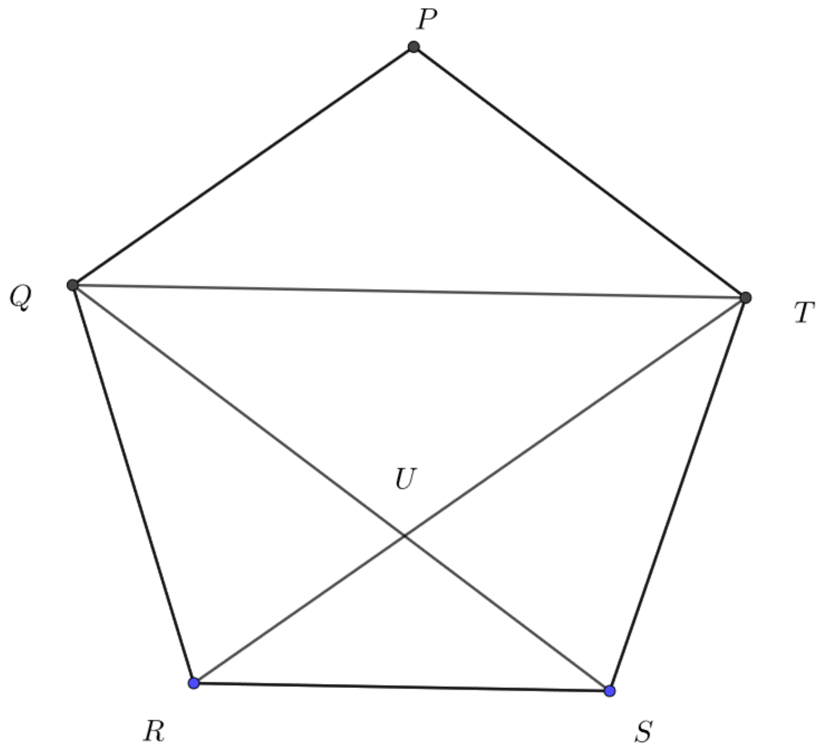
Each pentagon adjoins 5 hexagons; each hexagon adjoins 3 pentagons. Therefore $5p = 3h$, implying $h = 20$.

What we have shown so far is that a polyhedron made of 12 pentagons and 20 hexagons satisfies all the necessary conditions. But this isn't enough to guarantee existence; in other words we have not shown that 12 pentagons and 20 hexagons can actually be combined into a semi-regular polyhedron. However, if they can be so assembled, then we have found how many of each we need and this is what the question required.

6. A diagonal of a polygon is a line segment joining two vertices that is not a side of the polygon. Find the length of any diagonal of a regular pentagon of side 1, giving your answer in the form $a + b\sqrt{5}$, where a and b are rational.

Solution

First note that all the diagonals are the same length because each is the third side of an isosceles triangle having two sides of length 1 that include an angle of $\frac{3\pi}{5}$. (This could be proved formally by invoking the SAS criterion for congruent triangles.) The diagram below shows the pentagon and some of its diagonals. The diagonals shown have been chosen with the following considerations in mind. In the isosceles triangle TPQ the angle $\measuredangle TPQ$ is an internal angle of the polygon. Hence we can find it and we can also find the base angles $\measuredangle PQT$ and $\measuredangle QTP$. A simple trigonometrical calculation then gives the length of the diagonal QT but not in the form required. If we can find a pair of similar triangles, then the ratios involved might provide an answer in the desired form because no trigonometry will be involved.



Returning to triangle TPQ, simplify the notation by setting $\frac{\pi}{5} = \theta$. All the internal angles of the pentagon are then equal to 3θ. In particular $\measuredangle TPQ = 3\theta$ and so the base angles $\measuredangle PQT$ and $\measuredangle QTP$ are both equal to θ. Applying the same argument to the isosceles triangles QRS and RST shows that $\measuredangle SQR = \measuredangle STR = \theta$. Consideration of the angles around vertex Q now shows that $\measuredangle TQS = \theta$ and similarly, using the angles around vertex T, $\measuredangle RTQ = \theta$. Finally, by angles in a triangle applied to QUT, we have $\measuredangle QUT = 3\theta$. We can now see that triangles TPQ and QUT are similar and share the side QT. They are therefore congruent (ASA) so $QU = UT = 1$.

Now consider the triangle SUR which is plausibly similar to QUT; let us check this. The angle $\measuredangle SUR$ is vertically opposite to $\measuredangle QUT$ and so is equal to 3θ. Furthermore, consideration of the isosceles triangles QRS and RST shows that $\measuredangle RSQ = \measuredangle TRS = \theta$. Summing up, the triangle SUR is, like QUT, a 3θ, θ, θ triangle. They are therefore similar and we have

$$\frac{QU}{SU} = \frac{UT}{UR} = \frac{TQ}{RS}.$$

Now recollect that $QU = UT = 1$. Since triangle SUR is isosceles (equal base angles), it follows that $SU = UR$; let their common value be v. This in turn implies that the length of the diagonal SQ is $v + 1$. But TQ is another diagonal so $TQ = v + 1$. From the equal ratios we now have

$$\frac{1}{v} = \frac{v+1}{1}.$$

Multiplying up gives the quadratic equation $v^2 + v - 1 = 0$ with roots $v = \frac{-1\pm\sqrt{5}}{2}$. Since v is a length, it is clearly equal to $\frac{-1+\sqrt{5}}{2}$ and we conclude that the length of a diagonal is

$$v + 1 = \frac{1+\sqrt{5}}{2} = \frac{1}{2} + \frac{1}{2}\sqrt{5}$$

in the desired form.

Remark: You will recognise this number if you are familiar with the golden ratio. It appears in many interesting situations, so it is well worth exploring if you have not met it before. It is widely documented or could be an idea to discuss with your mentor.

7. Find all real values of x, y and z such that

$$(x+1)yz = 12 \quad (y+1)zx = 4 \quad \text{and} \quad (z+1)xy = 4.$$

[British Mathematical Olympiad Round 1 2008 Question 2]

Solution

The two 4's strongly suggest that we equate the left-hand sides of the second and third equations and then cancel the x's. However, for this to be valid requires $x \neq 0$. In fact none of the unknowns can be equal to 0: any zero value would give a contradiction in at least one of the equations.

Equating the left-hand sides of the second and third equations and then cancelling as described yields $(y+1)z = (z+1)y$, which in turn gives $y = z$. Substituting in the first two equations produces

$$(x+1)y^2 = 12 \quad \text{and} \quad (y+1)xy = 4.$$

Subtracting shows that $y^2 - xy = 8$ and it then follows that $xy = y^2 - 8$. Substituting for xy and z in the third equation gives $(y+1)(y^2-8) = 4$ or $y^3 + y^2 - 8y - 12 = 0$. It is clear that $y = 3$ is a root of this cubic and so the factor theorem implies that $y - 3$ is a factor of the left-hand side. Then the full factorisation is $(y-3)(y+2)^2 = 0$. Hence $y = 3$ or $y = -2$, which implies that $y = z = 3$ or $y = z = -2$.
When $y = z = 3$, then substituting in the first equation gives $x = \frac{1}{3}$.
When $y = z = -2$, then substituting in the first equation gives $x = 2$.

As usual, these solutions should be checked. Substituting the first set of values in the left-hand sides of the three original equations produces, respectively, $\frac{4}{3} \times 3 \times 3 = 12$, $4 \times 3 \times \frac{1}{3} = 4$ and $4 \times \frac{1}{3} \times 3 = 4$. Thus we have a valid solution. Checking the other set of values is left to the reader.

The full solution is therefore $x = 2$ and $y = z = -2$ or $x = \frac{1}{3}$ and $y = z = 3$.

8. A ***partition*** of a set S is a pair of sets A and B whose union is S and whose intersection is the empty set. Let $m \geqslant 3$ be an integer and let $S = \{3, 4, 5, \ldots, m\}$. Find the smallest value of m such that, for every partition of S into two subsets, at least one of the subsets contains integers a, b and c (not necessarily distinct) such that $ab = c$.

Solution

There does not appear to be an obvious way into this question. Also the last sentence, which mentions "every partition" and at "least one of the subsets" potentially refers to a very large numbers of sets. In a situation like this, the use of proof by contradiction can reduce the number of sets to be considered.

First let us get some idea of how big the required m might be. To use contradiction we need to assume that there is a partition of S into subsets T and U, say, such that neither contains a solution to the given equation. We also have to find candidates for a, b and c. An economical way to do this is to use just one element of S and consider its powers. To keep things simple we can try to use the smallest element of S, namely 3.

Without loss of generality assume that $3 \in T$. Then necessarily $3^2 \in U$. If $3^3 \in T$, then because $3^4 = 3 \times 3^3 = 3^2 \times 3^2$, one of the subsets must contain a solution to the given equation. Contradiction.

Now assume that $3^3 \in U$. Because $3^2 \in U$ we must have $3^4 = 3^2 \times 3^2 \in T$. But $3 \in T$ and so $3^5 = 3 \times 3^4 \in U$. But now U contains a solution to the equation because 3^2, 3^3 and 3^5 all belong to U. Contradiction. Thus our assumption is wrong and the required result holds for $m = 3^5 = 243$.

Of course we have not considered the possibility that the result holds for some value of $m \leqslant 242$. However, in that case, consider the partition given by $T = \{3, 4, 5, 6, 7, 8, 81, 82, \ldots, m\}$ and $U = \{9, 10, \ldots, 80\}$. Neither of these sets contains a solution to the equation. The last point to notice is that this example deals with many, though not all, values of $m \leqslant 242$. However, it can be adapted to cover the remaining values.

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

G. H. Hardy

Sheet 3

Solutions and comments

This programme of the Mentoring Scheme is named after G. H. Hardy (1877–1947).

See http://www-history.mcs.st-andrews.ac.uk/Biographies/Hardy.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

Mentoring Scheme, UK Mathematics Trust, School of Mathematics,
University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 mentoring@ukmt.org.uk www.ukmt.org.uk

Version 1.2, Nov 2020

1. Do the three hands on a clock (hour, minute and second hands) ever divide the clockface into three equal segments?

SOLUTION

Normally a time is specified by three pieces of information (seconds, minutes and hours) but to keep things simple we use a single variable and describe a time as h hours past noon. (Note that h does not have to be an integer.) In this time the hour hand will have moved through $30h$ degrees, the minute hand through $360h$ degrees, and the second hand through ____ degrees.

For the hands to divide the clock face into three equal segments we need the angle between any two hands to be ____ degrees. Applying this to the hours and minute hands, we see that the difference between the angles through which they have turned must be a multiple of ____. This implies

$$330h = 120a,$$

where a is an integer. Here a cannot be a multiple of 3, otherwise the hands would be on top of each other. Applying similar reasoning to the minute and second hands we find that

$$59 \times 360h = 120b$$

for an integer b which is not a multiple of 3. Now we can eliminate h to get a relationship between a and b:

$$\frac{b}{a} = \frac{59 \times 360}{330} = \frac{59 \times 12}{11},$$

so $59 \times 12a = 11b$. however, this means that b must be a multiple of ____. This is a contradiction so there is no time when the hands divide the clock face into three equal segments.

2. In the grid below w, x, y and z are four distinct, non-zero digits.

w	x
y	z

Find all values of w, x, y and z such that $10w + x + 10y + z + 10w + y + 10x + z = 200$.

SOLUTION

The starting point is

$$10w + x + 10y + z + 10w + y + 10x + z = 200. \tag{1}$$

Immediately note that the problem is symmetric in x and y: if they are interchanged , then (1) remains the same. On collecting like terms, (1) becomes

$$20w + 11(x + y) + 2z = 200 \tag{2}$$

and it is immediately obvious that $x + y$ must be even. Let $x + y = 2m$. Then, after cancelling and rearranging, (2) becomes

$$10w + z = 100 - 11m, \text{ where } x + y = 2m. \tag{3}$$

Now $10w + z$ is conveniently in the form of a two-digit number and the possible values that w and z can take are displayed in the table below. But first some comments on the entries in the table will be in order.

- In the fourth column possible values of x and y appear with $x < y$. But, in this context, see the comment after the table.
- The digits all have to be distinct and this excludes some choices. For example, in row B the choice $x = 2$, $y = 2$ is excluded because it is a repetition; in row D the x, y pairs 2, 6 and 3, 5 lead to repetitions once w and z are calculated.
- Since $10w + z = 100 - 11m$, then $10w + z - 1$ is divisible by 11. From this it follows that $-w + z - 1$ is divisible by 11. Since w and z are both between 1 and 9 (inclusive), it must be the case that $z = w + 1$.

Row	m	$x + y$	x, y ($x < y$)	$10w + z$	w	z	Comment
A	1	2		89	8	9	$x = y = 1$ not allowed
B	2	4	1, 3	78	7	8	$x = y = 2$ not allowed
C	3	6	1, 5 and 2, 4	67	6	7	
D	4	8	1, 7	56	5	6	$y = z = 6$ and $y = w = 5$ not allowed
E	5	10	1, 9 and 2, 8 and 3, 7	45	4	5	$x = w = 4$ not allowed
F	6	12	5, 7	34	3	4	$x = z = 4$ and $x = w = 3$ not allowed
G	7	14	5, 9 and 6, 8	23	2	3	
H	8	16	7, 9	12	1	2	
I	9	18		1	0	1	$x = y = 9$ not allowed

The fourth, sixth and seventh columns of the table give eleven possible sets of values of the variables. In each case $x < y$. As remarked above the symmetry of the situation means that swapping the values of x and y will lead to another set of solutions, making twenty-two in all.

3. You learnt about Euler's formula $F - E + V = 2$ in G H Hardy, Sheet 2, Question 5. This formula is also true in 2 dimensions. Imagine that a polyhedron is made of rubber. By the removal of one face, the rest of the 3-dimensional object can be pounded down to 2 dimensions, while preserving all the properties of edges and vertices. However, we have to remember that the "outside" is also a face. In 2-d we write the formula as $F - m + n = 2$ where m is the number of edges and n is the number of vertices. This is particularly useful when we study maps. Use these ideas to prove that in every map there is at least one contiguous region with five or fewer adjacent neighbours. (You can assume all vertices are of degree three, that is three edges meet at every vertex.)

Solution

If $n(i)$ represents the number of regions with exactly i neighbours and if every region has at least

six neighbours, then

$$F = \sum_{i=6}^{\infty} n(i) \quad \text{and} \quad 2m = \sum_{i=6}^{\infty} i \cdot n(i) \geqslant 6 \sum_{i=6}^{\infty} n(i) = 6F.$$

In other words, if each region on the map has at least six neighbours then $6F \leqslant 2m$ must be true. As we are allowed to assume that all vertices are of degree 3, then $3n = 2m$ must also be true. Substituting these two expressions into Euler's formula, we obtain

$$\frac{1}{3}m - m + \frac{2}{3}m \geqslant 2.$$

This is equivalent to $0 \geqslant 2$, which we know not to be true. Therefore $6F \leqslant 2m$ cannot hold and a region with five or fewer neighbours must be present.

4. In a playground there is a cubical climbing frame with each dimension being 3 metres. The cube is subdivided into unit twenty-seven sub-cubes of dimensions 1m x 1m x 1m. A child can enter from any of the four ground-level corner sub-cubes. A child is allowed to move from one sub-cube to another with which it shares a face. If a child wishes to make a tour of each of the twenty-seven sub-cubes, visiting each exactly once, after beginning at one of the sub-cubes in the corner of the big cube, prove that her tour cannot end at the central sub-cube.

SOLUTION

Solution A: You may want to start by checking that it is possible for all the sub-cubes to be visited exactly once. All the sub-cubes can be sorted into two sets such that, within each set, no two sub-cubes share a face. You can visualise this by the checker board colouring method. Every sub-cube should be assigned a different colour from the ones it shares a face with. Think of the twenty-seven sub-cubes as three layers of 3×3 sub-cubes stacked one on top of each other. This way you will end up with the top and bottom layers coloured identically and the middle layer having the opposite colouring. Therefore a child's movement will be alternating between these two sets. If we call the initial corner sub-cube the first sub-cube, then the last sub-cube to be visited, the twenty-seventh, must be in the same set as the first cube. But the sub-cube in the centre is not in the same set as the first cube.

Solution B: Coordinates (a, b, c) can be assigned to each sub-cube with a, b, c ranging from 1 to 3. For example one of the four ground level corner sub-cubes can be assigned coordinate $(1, 1, 1)$, the other three being $(1, 3, 1)$, $(3, 1, 1)$ and $(3, 3, 1)$. The other coordinates are assigned in the same way. Every time a child moves from one sub-cube to another, one of the coordinate value goes up or down by one. Therefore the total value of the coordinates varies between odd and even, each time a child makes a move. Any of the possible initial sub-cubes would have the sum of the coordinate values being odd, as we have listed them above. Therefore its total is odd. The centre sub-cube has coordinates $(2, 2, 2)$, its total value being even. The twenty-seventh sub-cube to be visited must have same parity in total coordinate value as the first sub-cube. Therefore the centre cube cannot be the last cube to be visited.

Remark: You may like to consider the points of similarity between the two solutions presented.

5. *Introductory note: The* ***binomial coefficient*** $\binom{n}{k}$ *is the number of ways of choosing k objects from a set of n (distinguishable) objects, without regard to order. For example* $\binom{4}{2} = 6$ *and* $\binom{5}{3} = 10$*. You may have met binomial coefficients already but you do not need any technical knowledge to answer this question. Focus on the statement that they can count the number of* ***choices*** *in a particular situation. For each part of the question try and think of a choosing process that will help you to answer it.*

In what follows $n \geqslant 1$.

a) Prove that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n.$$

b) Prove that, for $0 \leqslant k \leqslant n$,

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

c) Prove that

$$0 \times \binom{n}{0} + 1 \times \binom{n}{1} + 2 \times \binom{n}{2} + \cdots + n \times \binom{n}{n} = n2^{n-1}.$$

d) What is the average (mean) size of a subset of the set $N = \{1, 2, \ldots, n\}$?

Solution

We follow the advice given in the note: selecting a committee from a group of people is a time-honoured scenario.

a) In how many ways can we select a committee (of any size) from a class of n students?

If $0 \leqslant k \leqslant n$, then a committee of k students may be chosen in $\binom{n}{k}$ ways. Thus the sum on the left-hand side is the total number of ways of creating a committee of any size k. For the right-hand side, we look at the situation from a different point of view. Consider each of the n students in turn. There are two choices in each case: either the student is included in a given committee, or excluded. Since there are two possible choices or each student, there are 2^n possible committees. This establishes the identity.

b) We use the same situation as in part *a)* and could start from the idea that a student is picked out in some way. We could ask for the number of ways in which we can create a committee of size k from a class of n students, where one of the committee members is designated as the chair.

As before the number of different committees of size k is $\binom{n}{k}$. There are also k choices for the chair from among the students. Hence the number of different committee, chair pairs is $k\binom{n}{k}$. This takes care of the left-hand side. For the right-side, note that a chair can be chosen from the n students in n ways. For each possible chair, the number of ways of choosing the rest of the committee from the remaining $n-1$ students is $\binom{n-1}{k-1}$. So overall, the number of ways of selecting a committee, chair pair is $n\binom{n-1}{k-1}$. This establishes the identity.

Alternatively, if you are familiar with binomial calculations, then this is one case (because no

sum is involved) where the algebra is simple. We have

$$k\binom{n}{k} = \frac{k \times n!}{k! \times (n-k)!} = \frac{n!}{(k-1)! \times (n-k)!} = \frac{n \times (n-1)!}{(k-1)! \times ((n-1)-(k-1))!} = n\binom{n-1}{k-1},$$

as required.

c) Now the left-hand side of the identity in part *b)* appears as the general term in a sum. We can regard this sum as the number of ways of creating a committee (of any size) from a class of n students, where one of the committee members is designated as chair. For the expression on the right-hand side, first select the chair from the class of n students. Then from the remaining $n-1$ students, by the argument used in part *a)*, there are 2^{n-1} ways to choose a subset of them to form the rest of the committee. This establishes the identity.

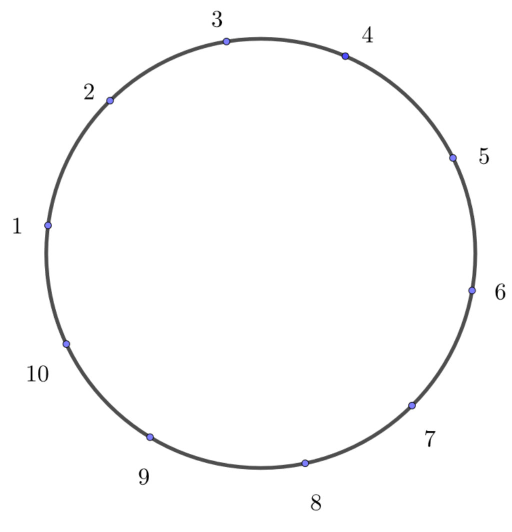
d) Now we have a set consisting of the first n positive integers, rather than a set of n students. The size of a subset of N is the number of elements in it. Further, N has $\binom{n}{k}$ subsets of size k. Hence the left-hand side of the identity in part *c)* is the sum of the sizes of all the subsets of N. Moreover, we now know that this is equal to $n2^{n-1}$. To obtain the average size of a subset we must divide $n2^{n-1}$ by the number of subsets of N. But this is 2^n by the method used in part *a)* to obtain the right-hand side of the identity. Hence the average size of a subset is

$$\frac{n2^{n-1}}{2^n} = \frac{n}{2}.$$

6. A bank has 1694 employees. They gather on a jubilee occasion and are all seated around one round table. It is known that the salaries of any two employees seated next to each other differ by £2 or £3. Find the maximum difference in salaries of two employees, if it is known that all the salaries are different.

Solution

You might find it helpful to consider a smaller bank first, say one with 10 employees. Below is a diagram showing them around their round table.



Suppose we start with employee 1, with salary $S(1) = 0$. The general picture must be that the employee with maximum difference in salary is somewhere over the other side of the table. We

could make the salaries go up in jumps of £3 and then decrease again. However, we would have to put in some jumps of £2 to avoid any equal salaries. A set of salaries satisfying the conditions is shown in the following table.

Employee k	1	2	3	4	5	6	7	8	9	10
Salary $S(k)$	0	2	5	8	11	14	12	9	6	3

The maximum salary difference is 14, between the salary of employee 6 and that of employee 1.

After that exploratory example, return to the bank with 1694 employees. Number them clockwise around the table; suppose employee 1 has the lowest salary and the maximum difference in salary is enjoyed by employee n. (In the example $n = 6$.) Denote the maximum salary difference by d. There are $n - 1$ intervals between employee 1 and employee n, so $d \leqslant 3(n-1)$. Completing the circle back to 1 there are $1694 - n + 1 = 1695 - n$ intervals, implying that $d \leqslant 3(1695 - n)$. Combining the two inequalities,

$$d \leqslant \frac{3(n-1+1695-n)}{2} = \frac{1694 \times 3}{2} = 847 \times 3 = 2541.$$

Thus we have found an upper bound for d but there is no guarantee that it can be achieved. In fact, it cannot be reached as that would imply that the difference between any two neighbours was 3 and hence the salaries would not all be distinct. However, we can construct a solution for 2540 based on the procedure in the example.

- Start with $S(1) = 0$, $S(2) = 2$.
- set $S(k) = S(k-1) + 3$ for $k = 3, 4, \ldots, 848$.
- $S(849) = S(848) - 2$.
- Set $S(k) = S(k-1) - 3$ for $k = 850, 851, \ldots, 1694$.

Then $S(848) - S(1) = 2540 - 0 = 2540$, so this value of d can be achieved.

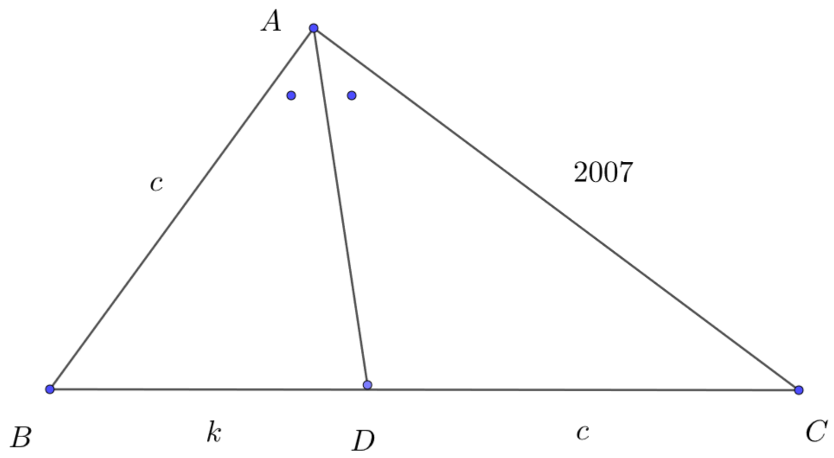
Remark: Where did 1694 come from? Note that the Bank of England was founded in 1694. However, the bank in the question is not intended to portray any institution whatsoever in the real world.

7. Triangle ABC has integer-length sides and $AC = 2007$. The internal bisector of $\measuredangle BAC$ meets BC at D. Given that $AB = CD$, determine AB and BC.

[British Mathematical Olympiad Round 2 2007 Question 1]

Solution

The configuration is as shown below.



Let $AB = CD = c$ and let $BD = k$. By the angle bisector theorem,

$$\frac{c}{k} = \frac{2007}{c}$$

and so

$$c^2 = 2007k = 3^2 \times 223 \times k.$$

Since the left-hand side is a square, the right-hand side must be a square also. Since 223 is prime it follows that $k = 223\lambda^2$ for some positive integer λ.

If $\lambda = 1$, then $k = 223$, $c^2 = 3^2 \times 223^2$ and $c = 3 \times 223 = 669$. Hence $AB + BC = 2c + k = 1561 < 2007 = AC$. This contradicts the triangle inequality.

If $\lambda = 2$, then $k = 892$, $c^2 = 3^2 \times 223^2 \times 2^2$ and $c = 3 \times 223 \times 2 = 1338$. Hence $AB = 1338$ and $BC = c + k = 2230$. These values certainly give a valid triangle.

If $\lambda > 2$, then $BC = k + c \geqslant 3^2 \times 223 + c = 2007 + c$. However, $AC + AB = 2007 + c$, implying that $BC \geqslant AC + AB$. This again contradicts the triangle inequality, so we conclude that the only possible case is $AB = 1338$ and $BC = 2230$.

8. Evaluate

$$\sum_{r=1}^{n} \frac{1}{\sin(2^r x)},$$

where x is a real number and $2^r x$ is not a multiple of π for $r = 1, 2, \ldots, n$.
Note: You may find it useful to recall the double-angle formulae. If you have not met them already, then note that they are $\cos 2x = \cos^2 x - \sin^2 x$ *and* $\sin 2x = 2\cos x \sin x$.

Solution

First note that the condition in the question avoids any case where $\sin(2^r x) = 0$. It is clear that something has to be done about 1 in the numerator, otherwise all the content of the expression is stuck in the denominator. Now 1 appears in the foundational trigonometric formula: "$\cos^2 + \sin^2 = 1$". It remains to select a useful angle.

Suppose we try $2^r x$. This is initially attractive as one factor $\sin 2^r x$ could be cancelled but overall we would be left with a difficult sum of terms. The presence of a factor 2 in each term

suggests that we might try to make use of double-angle formulae as well. Using this we have $\sin^2(2^{r-1}x) = \cos^2(2^{r-1}x) - \cos(2^r x)$. It follows that

$$1 = \cos^2(2^{r-1}x) + \sin^2(2^{r-1}x) = 2\cos^2(2^{r-1}x) - \cos(2^r x).$$

Then we can write

$$\frac{1}{\sin(2^r x)} = \frac{2\cos^2(2^{r-1}x)}{\sin(2^r x)} - \frac{\cos(2^r x)}{\sin(2^r x)} = \frac{2\cos^2(2^{r-1}x)}{2\sin(2^{r-1}x)\cos(2^{r-1}x)} - \frac{\cos(2^r x)}{\sin(2^r x)}.$$

Note the use of the double-angle formula for $\sin(2^r x)$ in the denominator of the first fraction. Upon cancelling, the previous line becomes

$$\frac{1}{\sin(2^r x)} = \frac{\cos(2^{r-1}x)}{\sin(2^{r-1}x)} - \frac{\cos(2^r x)}{\sin(2^r x)} = \cot(2^{r-1}x) - \cot(2^r x).$$

This shows we have a ***telescoping series*** since each term is of the form $f(r) - f(r-1)$. When n terms of such a series are added the sum becomes

$$f(1) - f(0) + f(2) - f(1) + \cdots + f(n-1) - f(n-2) + f(n) - f(n-1) = f(n) - f(0).$$

In our case $f(r) = -\cot(2^r x)$ and so

$$\sum_{r=1}^{n} \frac{1}{\sin(2^r x)} = \cot(2^0 x) - \cot(2^n)x = \cot x - \cot(2^n x).$$

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

G. H. Hardy

Sheet 4

Solutions and comments

This programme of the Mentoring Scheme is named after G. H. Hardy (1877–1947).

See http://www-history.mcs.st-andrews.ac.uk/Biographies/Hardy.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

Mentoring Scheme, UK Mathematics Trust, School of Mathematics,
University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 mentoring@ukmt.org.uk www.ukmt.org.uk

Version 1.2, Dec 2020

1. Evaluate
$$\sum_{n=1}^{\infty} \frac{n}{n^4+n^2+1}.$$

SOLUTION

It turns out that this question follows on nicely from the last question on the previous sheet. Could it be that this series telescopes as well? It could and indeed the method is always worth considering when the sum of a series is sought. To use the method, we need to split up the general term in some way. Note that the denominator has no linear factor because _____. (Fill in the reason for yourself.) A neat way of obtaining factors of higher degree is to use the fact that $(n^2+1)^2$ *almost* appears in the denominator. In fact

$$n^4+n^2+1 = n^4+2n^2+1-n^2 = (n^2+1)^2-n^2.$$

This difference of two squares factorises as $(n^2+1-n)(n^2+1+n)$. It is then easy to spot that

$$\frac{1}{n^2+1-n} - \frac{1}{n^2+1+n} = \frac{2n}{(n^2+1-n)(n^2+1+n)} = \frac{2n}{n^4+n^2+1}.$$

It follows that
$$\frac{n}{n^4+n^2+1} = \frac{1}{2(n^2+1-n)} - \frac{1}{2(n^2+1+n)}. \tag{*}$$

Unfortunately, what we want is for each term of the series to be of the form $f(n-1)-f(n)$ for some appropriate function f. A likely candidate is $f(n) = \dfrac{1}{2(n^2+1+n)}$. The corresponding $f(n-1)$ would then be $f(n-1) = \dfrac{1}{2((n-1)^2+1+(n-1))}$. However,

$$f(n-1) = \frac{1}{2((n-1)^2+1+(n-1))} = \frac{1}{2(n^2-2n+1+1+n-1)} = \frac{1}{2(n^2+1-n)}.$$

This last rational function is indeed the first term on the right-hand of (*). Therefore, for any positive integer N,

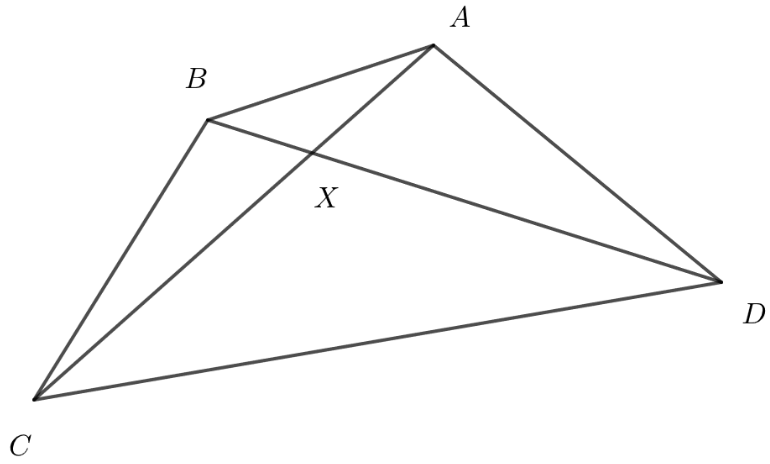
$$\sum_{n=1}^{N} \frac{n}{n^4+n^2+1} = \sum_{n=1}^{N}(f(n-1)-f(n)) = f(0)-f(N).$$

As $N \to \infty$, the denominator of $f(N)$ becomes very large and so $f(N)$ itself tends to zero. We conclude that
$$\sum_{n=1}^{\infty} \frac{n}{n^4+n^2+1} = f(0) = \frac{1}{2(0^2+1+0)} = \frac{1}{2}.$$

2. A field is in the shape of a convex quadrilateral*. The diagonals of the quadrilateral divide it into four triangles. Three of these triangles have areas of 400 m^2, 700 m^2 and 800 m^2. What is the largest area this field can have?
**For the definition of a convex polygon, see G H Hardy, Sheet 2, Question 3.*

Solution

Since the field is a convex quadrilateral, it has no angle greater than 180°. The layout is shown in the diagram below. In the solution the standard square brackets notation for an area is used: for example the area of a triangle UVW is usually denoted by $[UVW]$.



Consider triangles ABX and AXD. They can be considered as having the same height: the perpendicular from A onto BD. Therefore their areas are in the ratio of their bases and we have

$$\frac{[ABX]}{[AXD]} = \frac{BX}{XD}.$$

Considering triangles CXB and CDX in the same way, we have

$$\frac{[CXB]}{[CDX]} = \frac{BX}{XD}.$$

Equating the two expressions for $\dfrac{BX}{XD}$,

$$\frac{[ABX]}{[AXD]} = \frac{[CXB]}{[CDX]}.$$

We know the areas of three of the sections of the field, so it would be convenient for one of these fractions to have three values in the numerator. Rewrite the last equation as

$$[ABX] = \frac{[CXB][AXD]}{[CDX]} = \frac{[CXB][AXD][CDX]}{[CDX]^2}.$$

The areas in the numerator are 400, 700, 800 in some order; for convenience, put $[CDX] = a$. Then we have

$$[ABX] = \frac{400 \times 700 \times 800}{a^2}.$$

The area of the field will be largest when $[ABX]$ is largest and this will occur when $a = 400$. Then the fourth section will have area

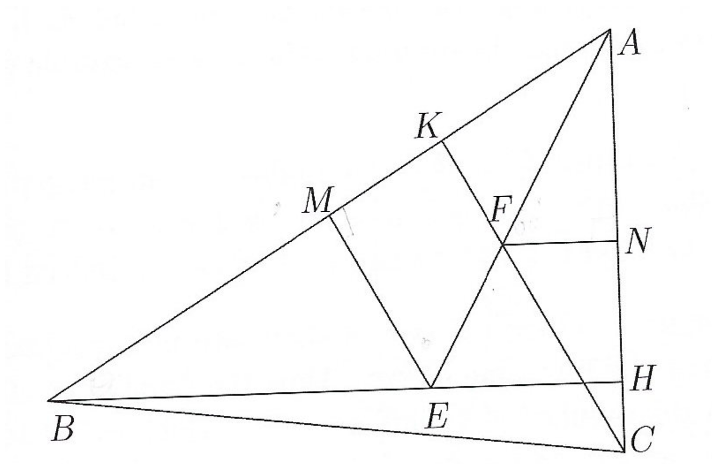
$$\frac{400 \times 700 \times 800}{400^2} = 1400.$$

The whole field then has area $400 + 700 + 800 + 1400 = 3300$. We conclude that the largest possible area of the field is 3300 m^2.

3. In triangle ABC the bisector of angle A, the perpendicular to side AB at its midpoint and the the altitude* from vertex B intersect in a single point. Prove that the bisector of angle A, the perpendicular to side AC at its midpoint and the altitude from vertex C also intersect in a single point.
*The **altitude** from any vertex of a triangle is the line from that vertex perpendicular to the opposite side.*

Solution

The configuration is shown in the diagram below.



Some preliminary comments: the conditions of the problem introduce a lot of constraints and you may wonder whether it is possible for such a configuration to exist. It may be reassuring to think of a equilateral triangle where a lot of special lines go through through the same "centre". For this problem, it is more then ever important to draw a large clear diagram to check that the requirements of the problem can be satisfied.

The diagram starts with the triangle ABC and BH is the altitude from vertex B perpendicular to AC. The line segment AE is the interior bisector of the angle at A, intersecting BH in E. The points M and N are the midpoints of AB and AC respectively. Then the conditions of the problem imply that ME is perpendicular to AB.

Next, let the perpendicular to AC through its midpoint N intersect AE in F. Produce (extend) CF to intersect AB in K. We shall show that CK is the altitude from C: this will be achieved by proving $\measuredangle CKA$ to be a right angle. First, the triangles BEM and MEA are congruent (SAS). This is because there are equal right angles at M and the corresponding sides including the right angles are equal. The reasons: ME is common to both triangles and $BM = MA$ because M is the midpoint of AB. It follows that $\measuredangle MBE = \measuredangle MAE$; denote their common value by θ. Then, since AE is an angle bisector, $\measuredangle HAE = \theta$.

Another pair of congruent triangles is MEA and HEA (AAS). Here the reasons for congruency are: $\measuredangle AME = \measuredangle AHE = 90°$; $\measuredangle MAE = \measuredangle HAE = \theta$; ME is common. We now have three congruent triangles, each of which contains one of the angles at E. Furthermore, each of these angles is equal to $90° - \theta$. Since they all lie on the line BH, each must be equal to $60°$. Hence $\theta = 30°$.

Now consider triangles CFN and AFN. They are congruent by (SAS). The reasons: FN is common; $AN = NC$ (N is the midpoint of AC) and the included angles $\measuredangle ANF$ and $\measuredangle CNF$ are

right angles. Hence $\measuredangle NCF = \measuredangle NAF = \theta = 30°$. Finally we examine the angles of the triangle AKC: $\measuredangle NCF = \measuredangle ACK = 30°$ and $\measuredangle CAK = 2\theta = 60°$. Hence $\measuredangle CKA = 90°$. In other words CK is the altitude from C, as required.

4. The French mathematician Joseph Fourier, who made signal advances in the application of mathematics to physical problems, was born in Burgundy in 1768. Find all polynomials $P(x)$ satisfying the equation

$$(x+1)P(x) = (x-1768)P(x+1).$$

Solution

Start by assuming that $P(x)$ is not identically zero. We are given that

$$(x+1)P(x) = (x-1768)P(x+1). \tag{1}$$

Setting $x = -1$ in (1) gives $0 = -1769P(0)$ and so $P(0) = 0$. It follows that x is a factor of $P(x)$ and we can therefore write $P(x) = xP_0(x)$, where $P_0(x)$ is a polynomial. Substituting in (1) yields

$$(x+1)xP_0(x) = (x-1768)(x+1)P_0(x+1). \tag{2}$$

Now set $x = 0$ in (2) to obtain $0 = -1768P_0(1)$ and so $P_0(1) = 0$. This in turn implies that $(x-1)$ is a factor of $P_0(x)$ and so we can write $P_0(x) = (x-1)P_1(x)$, where $P_1(x)$ is a polynomial. Hence

$$P(x) = xP_0(x) = x(x-1)P_1(x). \tag{3}$$

Again substituting from (3) in (2) gives

$$(x+1)x(x-1)P_1(x) = (x-1768)(x+1)xP_1(x+1). \tag{4}$$

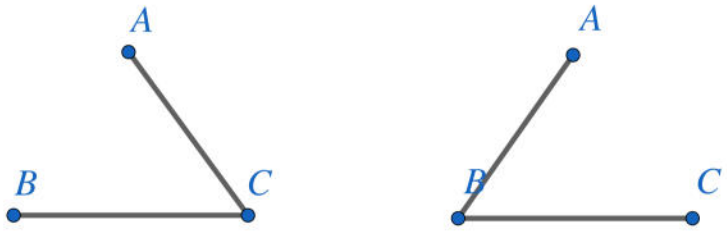
This process can be repeated by successively setting $x = 1, 2, 3, \ldots$ until we arrive at

$$(x+1)x(x-1)\cdots(x-1768)P_{1768}(x) = (x-1768)(x+1)x\cdots(x-1767)P_{1768}(x+1). \tag{5}$$

Now the linear factors on each side of (5) are the same and may be cancelled, since none of them is identically zero. This leaves us with $P_{1768}(x) = P_{1768}(x+1)$. Since this true for infinitely many real values of x, $P_{1768}(x)$ must be an arbitrary constant, precisely because $P(x)$ is a polynomial. (See the Remark below). Finally note that, if $P(x)$ were identically zero, then (1) would still be satisfied. We therefore conclude that $P(x)$ is an arbitrary constant, which may be zero.

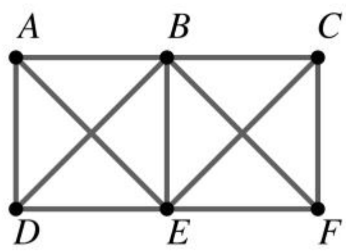
Remark: This deduction is not true for functions in general. You may like to think about this point or discuss it with your mentor.

5. You have met the concepts of graph and complete graph in *Question 8 of Hanna Neumann Sheet 4 and Question 4 of G H Hardy Sheet 2*. This question introduces to you the concept of a tree which is a connected graph that has no cycle. In a labelled graph the following are considered to be distinct trees:



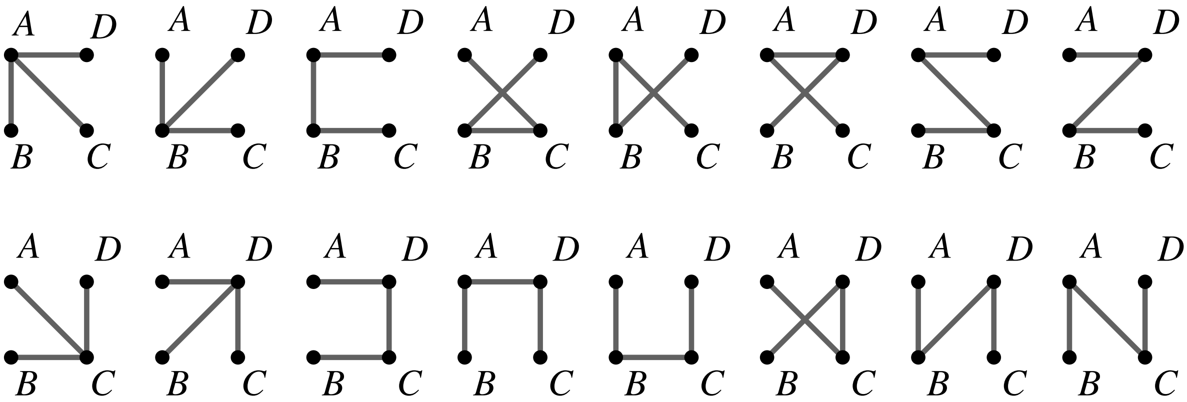
Furthermore, if AB were joined in the left-hand diagram, then a cycle would be formed and we would no longer have a tree.

- Sixteen distinct labelled trees can be found in the complete graph with four vertices. Can you draw them all?
- How many labelled trees can be found in the following graph?

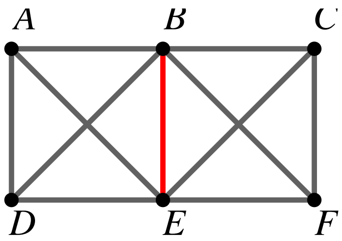


Solution

- The sixteen labelled trees with four vertices are shown below.



The graphs above are purposely arranged into two rows of eight trees. The graphs in one row have vertices C and D connected by an edge. The graphs in the other one do not.



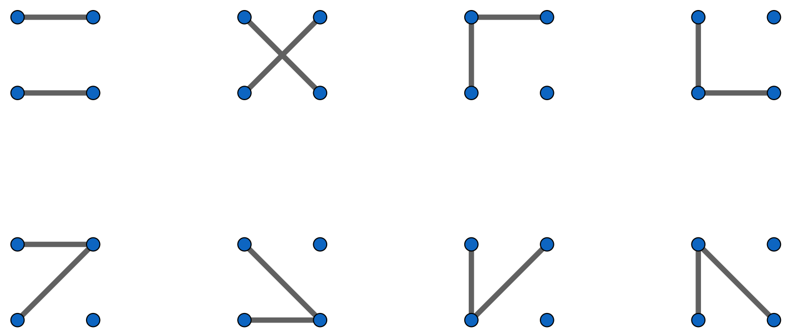
- The graph shown in the question is repeated above. It takes five edges to connect a tree of six vertices. (Why?) We are going to consider two cases: first, the edge BE (coloured red) is in the spanning tree (connecting all the vertices); second, the edge BE is not in the spanning tree.

  Case 1: the edge BE is in the spanning tree. In this case the points A, B, D and E are spanned by a sub-tree from the second row above. Likewise the points B, C, E and F are spanned by a sub-tree that is a reflection in a vertical line of one of those sub-trees from the second row. Therefore there are 8^2 possible trees of this kind.

  Case 2: the edge BE is not in the spanning tree.

  There are two cases to consider: the points A, B, D and E are spanned by a sub-tree; the points A, B, D and E are not spanned by a sub-tree.

  If there is a sub-tree that spans A, B, D and E, then it must be one of those from the first row above, giving 8 possibilities. The points C and F must be attached to the sub-tree that spans A, B, D and E by two edges. There are $\binom{5}{2} - 2 = 8$ ways of achieving this and they are shown below. In total therefore there are 8^2 trees of this kind.



Now suppose that there is no sub-tree that spans A, B, D and E. In this case there must be a sub-tree spanning the points B, E, C and F. (Why?) Then the argument in the previous paragraph, with minor obvious modifications, leads to the same conclusion.

Altogether there are $8^2 + 8^2 + 8^2 = 3 \times 8^2 = 192$ possible spanning trees of the graph.

6.

- Mrs Logistic went on holiday and bought two circular cake plates as a souvenir. These plates are particularly interesting because they help you to share the cake equally among your guests. The first plate helps you to share the cake among a maximum of twelve people. On it there are marks at the perimeter, each perpendicular to the tangent at the relevant point. By these marks there are some of the numbers ranging from 2 to 12; sometimes there can be just one number by itself. When Mrs Logistic has n guests, she cuts the cake from the centre to the marks where the number n is indicated. How many marks are there on this plate?
- The second plate is larger. The maximum sharing number is now sixteen. How many marks are on this plate?

  You may want to devise an exhaustive search to gain some initial insight into the problem. But ideally you should summarise your answer in the most concise manner possible.

SOLUTION

- First consider the **smaller plate** and note that the pieces cut for a given number of guests are to be equal. Also, though the marks on the plate are to be integers, fractions are used in the following solutions. They are converted into integers in an obvious way: for example the marks corresponding to the fractions with denominator 12 will all be labelled 12. Further, since $\frac{4}{12} = \frac{3}{9} = \frac{2}{6} = \frac{1}{3}$, the mark corresponding to $\frac{4}{12}$ will also be labelled 9 and 6 and 3.

  To cut a cake into 12 pieces, we need 12 marks $\left(\frac{0}{12} \text{ to } \frac{11}{12}\right)$.

  To cut a cake into 11 pieces, there need to be another 10 marks $\left(\frac{1}{11} \text{ to } \frac{10}{11}\right)$.

  To cut a cake into 10 pieces, we need another 8 marks. This is because we can omit $\frac{5}{10}$ from the list of marks from $\frac{1}{10}$ to $\frac{9}{10}$. And this in turn is because the $\frac{5}{10}$ mark has already

been provided by the $\frac{6}{12}$ mark.

To cut a cake into 9 pieces, we don't need marks for $\frac{0}{9}$, for $\frac{3}{9} = \frac{4}{12}$ or for $\frac{6}{9} = \frac{8}{12}$. Therefore we need another 6 marks, namely $\frac{1}{9}$, $\frac{2}{9}$, $\frac{4}{9}$, $\frac{5}{9}$, $\frac{7}{9}$ and $\frac{8}{9}$.

To cut a cake into 8 pieces, we don't need marks for $\frac{0}{8}$, for $\frac{2}{8} = \frac{3}{12}$, for $\frac{4}{8} = \frac{6}{12}$ or for $\frac{6}{8} = \frac{9}{12}$. Therefore we need another 4 marks, namely $\frac{1}{8}$, $\frac{3}{8}$, $\frac{5}{8}$ and $\frac{7}{8}$.

To cut a cake into 7 pieces, we need another 6 marks.

We need no more marks to cut a cake into 5, 4, 3 or 2 pieces since each of these numbers is a factor of one of the numbers considered previously.

Altogether the number of marks is $12 + 10 + 8 + 6 + 4 + 6 = 46$.

- For the **larger plate,** we reuse the marks above, plus the following.

  For 13 guests we need another 12 marks.

  For 14 guests, we need another 6 marks: $\frac{1}{14}$, $\frac{3}{14}$, $\frac{5}{14}$, $\frac{9}{14}$, $\frac{11}{14}$ and $\frac{13}{14}$. (Make sure that you understand why the other fractions are omitted.)

  For 15 guests we need another 8 marks: $\frac{1}{15}$, $\frac{2}{15}$, $\frac{4}{15}$, $\frac{7}{15}$, $\frac{8}{15}$, $\frac{11}{15}$, $\frac{13}{15}$ and $\frac{14}{15}$. (Check again.)

  Finally, by a now familiar argument, another 8 marks are needed to share a cake between 16 guests.

  Altogether we have $46 + 12 + 6 + 8 + 8 = 80$ marks for the larger plate.

Remark: The plates actually exist and were found in Brittany by the problem-setter.

7. The first term x_1 of a sequence is 2014. Each subsequent term of the sequence is defined in terms of the previous term. The iterative formula is

$$x_{n+1} = \frac{(\sqrt{2}+1)x_n - 1}{(\sqrt{2}+1) + x_n}.$$

Find the 2015th term x_{2015}.

[British Mathematical Olympiad Round 2 2015 Question 1]

Solution

To simplify the initial working, let $\sqrt{2} + 1 = k$ and let $\sqrt{2} - 1 = m$. Thus $km = 1$, $m = \dfrac{1}{k}$ and $k - m = 2$. Further note that

$$1 - m^2 = 1 - (3 - 2\sqrt{2}) = 2(\sqrt{2} - 1) = 2m. \tag{1}$$

Applying the iterative formula,

$$x_{n+1} = \frac{(\sqrt{2}+1)x_n - 1}{(\sqrt{2}+1) + x_n} = \frac{kx_n - 1}{k + x_n} = \frac{x_n - \frac{1}{k}}{\frac{x_n}{k} + 1} = \frac{x_n - m}{mx_n + 1}. \tag{2}$$

Therefore by (2)

$$x_{n+2} = \frac{x_{n+1} - m}{mx_{n+1} + 1} = \frac{\dfrac{x_n - m}{mx_n + 1} - m}{m\left(\dfrac{x_n - m}{mx_n + 1}\right) + 1} \tag{3}$$

Multiply top and bottom of this fraction by $mx_n + 1$ and use (1) to obtain

$$x_{n+2} = \frac{x_n - m - m^2 x_n - m}{mx_n - m^2 + mx_n + 1} = \frac{(1 - m^2)x_n - 2m}{2mx_n + (1 - m^2)} = \frac{x_n - 1}{x_n + 1}. \tag{4}$$

It follows that

$$x_{n+4} = \frac{x_{n+2} - 1}{x_{n+2} + 1} = \frac{\dfrac{x_n - 1}{x_n + 1} - 1}{\dfrac{x_n - 1}{x_n + 1} + 1} = \frac{-2}{2x_n} = \frac{-1}{x_n}.$$

Consequently

$$x_{n+8} = \frac{-1}{x_{n+4}} = \frac{-1}{\dfrac{-1}{x_n}} = x_n.$$

The calculations above show that the terms repeat in blocks of 8. This means that $x_{2015} = x_7$ because $2015 = 8 \times 251 + 7$.

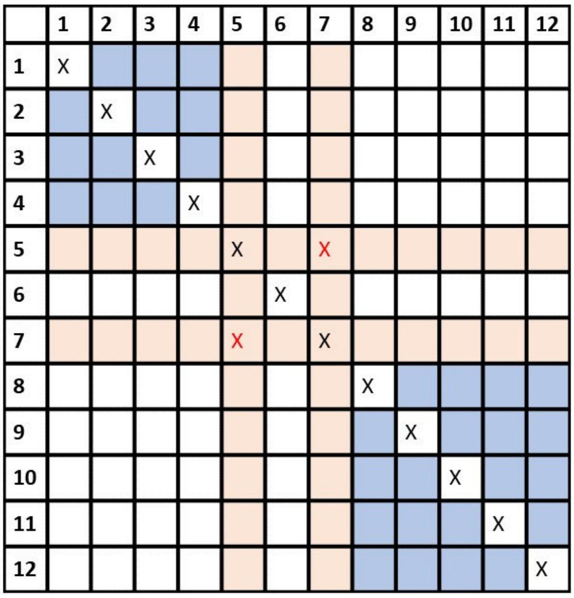
Now $x_9 = x_1 = 2014$ and by (4) $x_9 = \dfrac{x_7 - 1}{x_7 + 1}$; thus $2014 = \dfrac{x_7 - 1}{x_7 + 1}$.

This gives $2014x_7 + 2014 = x_7 - 1$ and so $x_{2015} = x_7 = -\dfrac{2015}{2013}$.

8. The 52 cards in a pack are numbered $1, 2, \ldots, 52$. Each of Anna, Boris, Cyril and Duscha picks a card from the pack, without replacement and with each card being equally likely to be chosen. The two people with the lower numbered cards form a team and the two people with the higher numbered cards form another team. Let $P(a)$ be the probability that Anna and Duscha are on the same team, given that Anna picks one of the cards from the pair a and $a + 9$ and that Duscha picks the other member of the pair. Find the minimum value of $P(a)$ for which $P(a) \geqslant \frac{1}{2}$.

Solution

The wording of the question is rather dense, so let us examine a simpler situation to appreciate what is going on. The diagram shows the case of a pack of twelve cards where $a = 5$ and Anna and Duscha each pick a card from the pair a and $a + 2$. The grid shows all the possible choices for Boris and Cyril. Suppose that Boris's choices are along the top row and Cyril's are down the left-hand column. For example, the square in column 2 and row 3 corresponds to Boris picking 2 and Cyril picking 3. The black X's down the main diagonal indicate that the cards are picked without replacement. The two red crosses indicate the $\{5, 7\}$ pair picked by Anna and Duscha. Since we want Anna and Duscha to end up on the same team, we want Boris and Cyril to pick squares up in the top left-hand corner or down in the bottom right-hand corner; these squares are coloured blue. The beige area consists of excluded choices involving cards 5 or 7.



The available squares, without X's or beige shading, are 90 in number. There are 32 blue squares. Hence in this case $P(a) = \frac{32}{90} = \frac{16}{45}$. However, it will be useful to recalculate this number in a way that can be adapted to the original problem. Once cards 5 and 7 have been removed there are 10 left in the small pack.

P(Boris and Cyril have the lower numbered cards) $= \dfrac{\binom{4}{2}}{\binom{10}{2}} = \dfrac{6}{45}$.

P(Boris and Cyril have the higher numbered cards) $= \dfrac{\binom{5}{2}}{\binom{10}{2}} = \dfrac{10}{45}$.

Hence P(Anna and Duscha are on the same team) $= \frac{16}{45}$, corresponding to the diagram. Now we can return to the original problem, remembering that Boris and Cyril will have 50 cards to choose from.

If Boris and Cyril pick a pair of lower-numbered cards, then these must be in the range 1 to $a - 1$. This probability is $\dfrac{\binom{a-1}{2}}{\binom{50}{2}} = \dfrac{(a-1)(a-2)}{50 \times 49}$.

If Boris and Cyril pick a pair of higher-numbered cards, then these must be in the range $a + 10$ to 52. This range contains $43 - a$ cards so the probability is $\dfrac{\binom{43-a}{2}}{\binom{50}{2}} = \dfrac{(43-a)(42-a)}{50 \times 49}$. Hence

$$P(a) = \frac{(a-1)(a-2) + (43-a)(42-a)}{50 \times 49}.$$

The numerator is $a^2 - 3a + 2 + 258 \times 7 - 85a + a^2 = 2(a^2 - 44a + 904)$. It follows that $P(a) = \dfrac{a^2 - 44a + 904}{25 \times 49}$ and so we require

$$P(a) = \frac{a^2 - 44a + 904}{25 \times 49} \geqslant \frac{1}{2}.$$

This can be rearranged as

$$(a-22)^2 - 484 + 904 \geqslant \frac{25 \times 49}{2} \quad \text{or} \quad (a-22)^2 \geqslant \frac{25 \times 49}{2} - 420 = \frac{1225 - 840}{2} = \frac{385}{2}.$$

Because a is an integer, it follows that $a - 22 \geqslant 14$ or $a - 22 \leqslant -14$; that is $a \geqslant 36$ or $a \leqslant 8$. Thus the minimum possible value of $P(a)$ is given by

$$P(8) = P(36) = \frac{64 - 352 + 904}{25 \times 49} = \frac{616}{25 \times 49} = \frac{88}{25 \times 7} = \frac{88}{175}.$$

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

G. H. Hardy

Sheet 5

Solutions and comments

This programme of the Mentoring Scheme is named after G. H. Hardy (1877–1947).

See http://www-history.mcs.st-andrews.ac.uk/Biographies/Hardy.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

Mentoring Scheme, UK Mathematics Trust, School of Mathematics, University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 mentoring@ukmt.org.uk www.ukmt.org.uk

Version 2.1, Jan 2021

1. Anne's number is obtained by writing down 20 consecutive positive integers, one after another in an arbitrary order. Peter's number is obtained in the same way, but with 21 consecutive positive integers. Can they obtain the same number?

Solution

One possibility for Anne's string of numbers is

$$4, \quad 5, \quad 6, \quad \ldots\ldots, \quad 20, \quad 21, \quad 22, \quad 23.$$

Then Peter's string could be

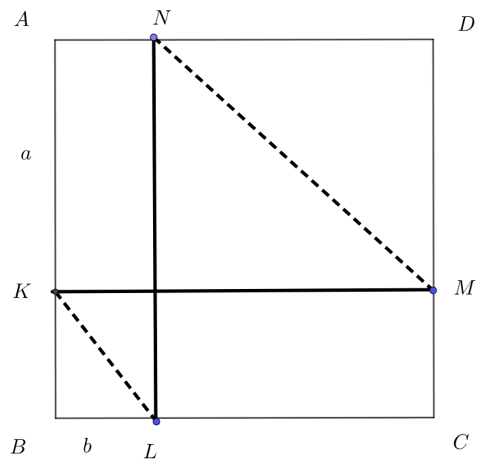
$$4, \quad 5, \quad 6, \quad \ldots\ldots, \quad 20, \quad 21, \quad 22, \quad 2, \quad 3.$$

These both provide the same integer $456\ldots20212223$.

2. The square $ABCD$ has sides of unit length. The points K, L, M and N lie on the sides AB, BC, CD and DA, respectively, of the square. They are such that KM is parallel to BC and LN is parallel to AB. The perimeter of triangle KBL is equal to 1. What is the area of triangle MND?

Solution

As shown in the diagram below, denote the lengths of AK and BL by a and b respectively. Now because KM and LN are parallel to adjacent sides of the square, it follows that $AKMD$ and $NLCD$ are parallelograms. (Of course, they are actually rectangles.) It follows that $MD = AK = a$ and $DN = LC$. However, since $ABCD$ is a unit square, $BL = b$ implies that $LC = 1 - b$ and so $DN = 1 - b$.



Now consider triangle KBL. Since the square has side 1, the length of KB is $1 - a$. Again, since the perimeter of the triangle is 1, it follows that the length of LK is $a - b$. Applying Pythagoras' theorem to the right-angled triangle KBL yields $(1 - a)^2 + b^2 = (a - b)^2$. On expansion this is

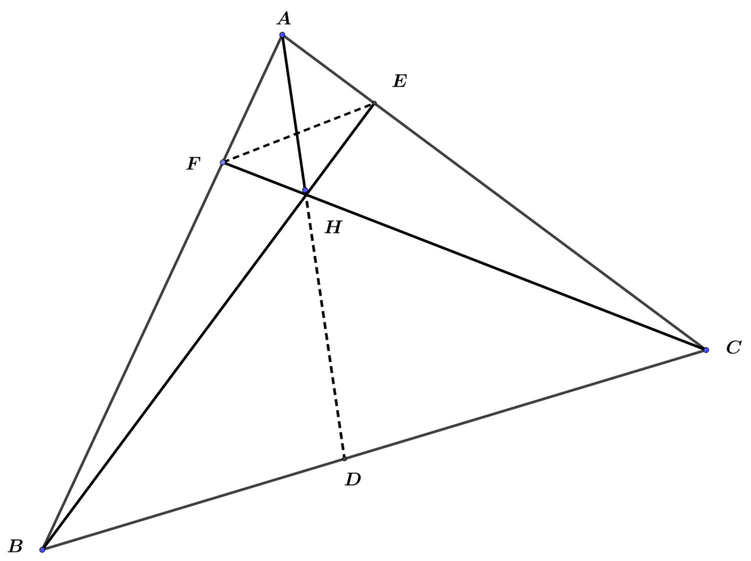
$$1 - 2a + a^2 + b^2 = a^2 - 2ab + b^2 \quad \text{or} \quad 1 = 2a(1 - b). \qquad (*)$$

We have already remarked that $MD = a$ and $DN = 1 - b$. Therefore the area of the triangle MND is $\frac{1}{2}a(1 - b)$. However, by (*) this is $\dfrac{1}{2} \times \dfrac{1}{2}$ and so the required area is $\dfrac{1}{4}$.

3. An ***altitude*** of a triangle is the line segment from a vertex perpendicular to the opposite side. Prove that the three altitudes of a triangle intersect in a point.
Note: this point is called the ***orthocentre*** *of the triangle.*

SOLUTION

The diagram below illustrates the configuration.



In the diagram BE and CF are the altitudes from B and C respectively. Denote their point of intersection by H. Join E and F and also join A to H, extending the segment AH to meet BC in D. Set $\measuredangle BAD = \theta$.

First observe that $\measuredangle AFC$ and $\measuredangle AEB$ are right angles. Hence $AFHE$ is a cyclic quadrilateral by the converse of the opposite angles result. This in turn implies

$$\measuredangle FEB = \measuredangle FEH = \measuredangle FAH = \measuredangle BAD = \theta$$

by angles in the same segment.

Next observe that $\measuredangle CEB$ and $\measuredangle BFC$ are both right angles. Hence $BCEF$ is a cyclic quadrilateral by the converse of angles in the same segment. It follows that

$$\theta = \measuredangle FEB = \measuredangle FCB = \measuredangle HCD,$$

again by angles in the same segment.

Now return to triangle FAH. It is a right-angled triangle with $\measuredangle FAH = \theta$. Hence $\measuredangle FHA = 90° - \theta$ and consequently $\measuredangle DHC = 90° - \theta$ by vertically opposite angles.

Finally consider triangle DHC, where we have $\measuredangle HCD = \theta$ and $\measuredangle DHC = 90° - \theta$. Therefore the third angle $\measuredangle CDH$ is a right angle. However, $\measuredangle CDH = \measuredangle CDA$ and so we see that AD is perpendicular to BC, implying that AD is the altitude from A. This proves the required result since AD passes through H.

4. The real numbers a and b are both positive. Find a necessary and sufficient condition for the equation

$$\sqrt{x+a} - \sqrt{x} = b$$

to have a solution for x. Write down this solution, given that the condition is satisfied.

Solution

In the search for x, some squaring must be carried out and it is advantageous to rearrange the equation as $\sqrt{x+a} = \sqrt{x} + b$. Then, after squaring, just one root will appear and in fact we have

$$x + a = x + 2b\sqrt{x} + b^2.$$

Since $b > 0$, this yields $\sqrt{x} = \dfrac{a-b^2}{2b}$. It will be immediately observed that, since $\sqrt{x}$ is non-negative, this statement is valid only if $a \geqslant b^2$. We therefore assume that this inequality holds.

Since $\sqrt{x} = \dfrac{a-b^2}{2b}$, it follows that $x = \dfrac{(a-b^2)^2}{4b^2}$. The argument so far has shown that, if x is to be a solution of the original equation, then the only possible candidate for x is $\dfrac{(a-b^2)^2}{4b^2}$. However, there is no guarantee that this candidate actually ***is*** a solution. It is essential to check that this value of x satisfies the equation. To this end, we substitute it into the left-hand side. First note that $x = \dfrac{(a-b^2)^2}{4b^2}$ implies that

$$x + a = \frac{(a-b^2)^2}{4b^2} + a = \frac{a^2 - 2ab^2 + b^4 + 4ab^2}{4b^2} = \frac{a^2 + 2ab^2 + b^4}{4b^2} = \left(\frac{a+b^2}{2b}\right)^2.$$

Further, since $\dfrac{a+b^2}{2b} > 0$, $\sqrt{x+a} = \dfrac{a+b^2}{2b}$; we also have $\sqrt{x} = \dfrac{a-b^2}{2b}$. Substituting in the left-hand side of the equation,

$$\sqrt{x+a} - \sqrt{x} = \frac{a+b^2}{2b} - \frac{a-b^2}{2b} = b,$$

i.e. the equation is satisfied.

Alternatively, if $a < b^2$, then $\sqrt{x}$, which has to be non-negative, is equal to $\dfrac{b^2-a}{2b}$. Then the substitution yields

$$\sqrt{x+a} - \sqrt{x} = \frac{a+b^2}{2b} - \frac{b^2-a}{2b} = \frac{a}{b}.$$

In this case the only possible candidate for a solution does not satisfy the equation. The only way that the candidate *could* provide a solution would be if we had $\dfrac{a}{b} = b$. However, this would imply $a = b^2$, which contradicts $a < b^2$.

Thus finally we can state that the necessary and sufficient condition for the equation to have a solution is $a \geqslant b^2$. When this condition is satisfied, the solution is $x = \dfrac{(a-b^2)^2}{4b^2}$.

Remark: If this situation seems strange, then it may help to consider a couple of numerical cases. If $a = 2$ and $b = 1$, then the condition is satisfied and the solution is $x = \frac{1}{4}$. If $a = 1$ and $b = 10$,

then the condition is not satisfied. Furthermore, the equation is $\sqrt{x+1} - \sqrt{x} = 10$. *It seems unlikely that the difference in the roots could be as large as 10. Thus intuitively we would not expect a solution in this case.*

5. *Recall that, for any real number x,* $\lfloor x \rfloor$ *is defined to be the greatest integer less than or equal to x. Thus* $\lfloor 2 \rfloor = 2$, $\lfloor \pi \rfloor = 3$ *and* $\lfloor -1.5 \rfloor = -2$.
Find the number of positive integers less than 1000 for which there exists a positive real number x such that $n = x\lfloor x \rfloor$.

Solution

It may seem that there is not much to get hold of in this question so an example is one place to start. Note that, if n has the required property, then x must be rational. Suppose that $x = \frac{5}{2}$. Then certainly $2 \leqslant x < 3$ and so $\lfloor x \rfloor = 2$. Consequently $x\lfloor x \rfloor = 5$, which is an integer and an integer that is greater than 2^2. Might examining squares be useful in general?

Suppose that the positive integer n has the form $x\lfloor x \rfloor$ for some real x. To simplify the notation, put $\lfloor x \rfloor = N$. Then $N \leqslant x < N + 1$ and $n = xN$. Since N must be positive, it follows that $N^2 \leqslant xN < N(N+1)$, that is $N^2 \leqslant n < N(N+1)$.

The next question to ask is: does ***every*** integer n in that range have the required property? Suppose that N and n are positive integers with $N^2 \leqslant n < N(N+1)$. Then, since $N > 0$, $N \leqslant \frac{n}{N} < N + 1$. Setting $\frac{n}{N} = x$. the preceding inequalities may be written as $N \leqslant x < N + 1$ and this implies $\lfloor x \rfloor = N$. Hence $\frac{n}{\lfloor x \rfloor} = x$ and multiplying up yields $n = x\lfloor x \rfloor$. This argument shows that, for any positive integer N, the integers from N^2 up to $N(N+1) - 1$ inclusive all have the desired property.

Therefore, to finish the problem we need to count the number of integers less than 1000 in ranges of this type. Now $32^2 = 1024$ and $31 \times 32 = 992$. Thus N takes values from 1 to 31. However, for each such N, there are N integers between N^2 and $N(N+1) - 1$ inclusive. We therefore require

$$1 + 2 + 3 + \cdots + 30 + 31 = \frac{31 \times 32}{2} = \frac{992}{2} = 496.$$

We conclude that there are 496 integers less than 1000 with the required property.

6. In a bag, I have slips of paper with the numbers 1 to 2000 written on them. I pick out four slips, one at a time, without replacement. What is the probability that the numbers emerge in increasing order?

Solution

We can calculate the number of ways to pick slips out of the bag. Often for this type of calculation, it's a good idea to proceed one step at a time. So there are 2000 ways to pick the *first* slip. Now, whatever we chose first, we cannot choose it again, so there are 1999 slips which we might pick second, for *every* choice of the first slip. So there are 2000×1999 ways to pick the first two slips. Extending this argument, there are

$$2000 \times 1999 \times 1998 \times 1997 \tag{1}$$

ways to pick the four slips without replacement.

Ideally, we would now calculate the number of ways to pick slips out of the bag *in increasing order*, and then divide this answer by the total number of ways of picking four slips. But it is hard to count these good ways in the same way that we arrived at the total number. There are still 2000 ways to choose the first slip, but then the number of ways to choose the second slip depends on the value of the first slip, and the complications grow for further slips.

So we need a different approach. Note that among the $2000 \times \ldots \times 1997$ ways to pick the slips are the orderings $(17, 1, 3, 2000)$, and $(3, 2000, 1, 17)$ and $(1, 3, 17, 2000)$. In fact, there are exactly $4 \times 3 \times 2 \times 1 = 24$ orderings which feature these four numbers, and of these 24 orderings, exactly one is in increasing order, namely $(1, 3, 17, 2000)$.

This argument generalises to sets of numbers other than $\{1, 3, 17, 2000\}$ of course! So we argue that we can divide all the orderings into groups of 24 orderings, where in each group, exactly one is in increasing order. Hence the number of orderings with this property is

$$\frac{2000 \times 1999 \times 1998 \times 1997}{24}. \tag{2}$$

To find the probability we divide the number of good orderings by the total number of orderings. So the probability that the numbers emerge in increasing order is $1/24$.

Note 1: We might be worried that the quantity in (2) *is not an integer. But because our argument shows that this quantity counts something (the number of orderings with a particular property), we have actually* proved *that it is an integer, without needing to use divisibility or arguments using factorials.*
Note 2: If you have come across the concept of ***conditional probability****, then you might choose to phrase the argument in that way instead.*
Note 3: Do you think that there is anything special about the value 2000? Which other integers could be used?

7. Long John Silverman has captured a treasure map from Adam McBones. Adam has buried the treasure at the point (x, y) with integer coordinates (not necessarily positive). He has indicated on the map the values of $x^2 + y$ and $x + y^2$, and these numbers are distinct. Prove that Long John has to dig only in one place to find the treasure.

[British Mathematical Olympiad Round 1 2009 Question 6]

Solution

The question asks us to show that the equations $x^2 + y = a$ and $x + y^2 = b$ can have at most one solution in integers if a and b are distinct. Suppose not, so that there are two different solutions (x_1, y_1) and (x_2, y_2). Then we have $x_1^2 + y_1 = x_2^2 + y_2$ and $x_1 + y_1^2 = x_2 + y_2^2$, giving

$$x_1^2 - x_2^2 = (x_1 - x_2)(x_1 + x_2) = y_2 - y_1, \tag{1}$$

$$y_1^2 - y_2^2 = (y_1 - y_2)(y_1 + y_2) = x_2 - x_1. \tag{2}$$

Multiplying these and factorising, we get

$$(x_1 - x_2)(x_1 + x_2)(y_1 - y_2)(y_1 + y_2) = (y_2 - y_1)(x_2 - x_1) = (x_1 - x_2)(y_1 - y_2). \tag{3}$$

Now, if $x_1 = x_2$, then it is easy to see from (1) and (2) that $y_1 = y_2$. However, in that case the solutions would not be different. Therefore we can assume that $x_1 \neq x_2$ and $y_1 \neq y_2$. This allows us to cancel factors $(x_1 - x_2)$ and $(y_1 - y_2)$ in equation (3). This yields

$$(x_1 + x_2)(y_1 + y_2) = 1.$$

Because all the variables are integers, it follows that either

$$x_1 + x_2 = y_1 + y_2 = 1. \tag{4}$$

or

$$x_1 + x_2 = y_1 + y_2 = -1. \tag{5}$$

Substituting from (4) into (1) gives

$$x_1 - x_2 = y_2 - y_1. \tag{6}$$

Substituting from (5) into (1) gives

$$-(x_1 - x_2) = x_2 - x_1 = y_2 - y_1. \tag{7}$$

Now consider (4) and (6), namely

$$x_1 + x_2 = y_1 + y_2 \quad \text{and} \quad x_1 - x_2 = y_2 - y_1.$$

Adding this pair of equations yields $x_1 = y_2$ and, in turn, $x_2 = y_1$. Now return to the original pair of equations and make these substitutions. The pair can be written as

$$x_1^2 + y_1 = x_1^2 + x_2 = a \quad \text{and} \quad x_2 + y_2^2 = x_2 + x_1^2 = b.$$

But this implies that $a = b$, contradicting our requirement that a and b be distinct. A similar contradiction arises if we use equations (5) and (7). Hence we are forced to conclude that there is just one solution to the original equations. For Long John this means that, once he has solved the equations, he has to dig in just one place to find Adam's treasure.

8. ***An exploration of powers in modular arithmetic.*** *For this question it will be useful to recall Fermat's Little Theorem, which appeared in Hanna Neumann, Sheet 5, Question 3. It states that, if p is a prime and a is a positive integer that is coprime to p, then $a^{p-1} \equiv 1 \pmod{p}$.*

a) Prove that if p and q are distinct primes then

$$x^{(p-1)(q-1)} \equiv 1 \pmod{pq} \tag{*}$$

provided that x is not a multiple of p and x is not a multiple of q.

b) Show that if the integers a and b satisfy $ab \equiv 1 \pmod{(p-1)(q-1)}$ and if $y \equiv x^a \pmod{pq}$, then it follows that $x \equiv y^b \pmod{pq}$.

c) The integers a and b are as in part *b*). If $a = 31$ and $n = pq = 15$, then find a value for b.

d) Staying with the situation in parts *b*) and *c*), let $a = 31$ and $x = 2$. Find the remainder r of 2^{31} on division by 15. Setting $y = r$, verify that $y^b \equiv 2 \pmod{15}$, where b is the value that you found in part *c*).

Solution

a) The algebra in this question can be simplified by the use of modular arithmetic. This appears, for example, in some Cartwright sheets and in Questions 2 and 3 of Hanna Neumann, Sheet 4. If you have not met it in these contexts, then sources of information are widely available or you can consult your mentor.

By Fermat's Little Theorem $x^{p-1} \equiv 1 \pmod p$ and $x^{q-1} \equiv 1 \pmod q$. By the familiar rules of modular arithmetic, we then have

$$(x^{p-1})^{q-1} \equiv 1^{q-1} \equiv 1 \pmod p \quad \text{and} \quad (x^{q-1})^{p-1} \equiv 1^{p-1} \equiv 1 \pmod q.$$

These equivalences can be written more simply as

$$x^{(p-1)(q-1)} \equiv 1 \pmod p \quad \text{and} \quad x^{(p-1)(q-1)} \equiv 1 \pmod q.$$

It follows that $x^{(p-1)(q-1)} = sp + 1$ and $x^{(p-1)(q-1)} = tq + 1$, for some non-negative integers s and t. Therefore

$$(p+q)x^{(p-1)(q-1)} = p(tq+1) + q(sp+1) = (p+q) + (s+t)pq.$$

Rewriting this as a congruence yields

$$(p+q)x^{(p-1)(q-1)} \equiv (p+q) \pmod{pq}.$$

We now observe that $p+q$ and pq are coprime and consequently $p+q$ may be cancelled to give $x^{(p-1)(q-1)} \equiv 1 \pmod{pq}$, as required.

b) Now we use (*) to prove this next result: start with $y \equiv x^a \pmod{pq}$. Raising both sides to the power b gives us $y^b \equiv x^{ab} \pmod{pq}$. Since $ab \equiv 1 \pmod{(p-1)(q-1)}$, there must be a non-negative integer k such that $ab = k(p-1)(q-1) + 1$. Hence

$$y^b = x^{ab} = x^{k(p-1)(q-1)+1} = (x^{(p-1)(q-1)})^k \times x.$$

Using (*) it follows that

$$y^b \equiv 1^k \times x \equiv x \pmod{pq}$$

and so the proof of part *b*) is complete.

c) In this simple case, we do not have much choice in factorising n as as the product of two primes: let us say that $p = 3$ and $q = 5$. Then $(p-1)(q-1) = 8$. Following part *a),* we seek b such that $31b \equiv 1 \pmod 8$. Some sensible guesswork is useful here: go up the 31-times table searching for products that are 1 more than a multiple of 8. We find that $7 \times 31 = 217 = 216 + 1$ and $216 = 6^3$, which is divisible by 8. Thus we can take $b = 7$.

d) We have

$$2^{31} \equiv (2^5)^6 \times 2 \equiv (2 \times 15 + 2)^6 \times 2 \equiv 2^6 \times 2 \equiv 128 \equiv 8 \pmod{15}.$$

Thus the remainder of 2^{31} on division by 15 is 8. Now consider $y^b \equiv 8^7 \pmod{15}$. We have $8^7 = 2^{21} = (2^7)^3 = (128)^3$ and we already know that $128 \equiv 8 \pmod{15}$. Therefore

$$x \equiv 8^7 \equiv 8^3 \equiv 2^9 \equiv 512 \equiv 34 \times 15 + 2 \equiv 2 \pmod{15}.$$

This is the required result.

Note 1: This question may appear to be just an exercise in manipulating powers in modular arithmetic. However, it is related to the calculations used in RSA cryptography. If you look back to part b), you will see that x and y are closely linked by the values of a and b. You could imagine that one is a coded version of the other and we have a procedure for moving between them. We started with $x = 2$, then "coded" it as $y = 8$, using a. Then, using b, we could get back from y to x. Note that finding suitable values for a and b depended on factorising 15, a trouble-free process. When RSA is used in what is usually referred to as the real world, cryptographers choose very large primes p and q and use their product n. To an eavesdropper, n is a huge number that cannot feasibly be split into two prime factors. Since the decryption depends on knowing this factorisation, any message remains secure, for the present at least!
Note 2: In our simple example, it was easy to spot a value for b. However, in subsequent sheets, we will meet a method that can be relied on to calculate b in all circumstances.

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by Oxford Asset Management

G. H. Hardy

Sheet 7

Solutions and comments

This programme of the Mentoring Scheme is named after G. H. Hardy (1877–1947).

See http://www-history.mcs.st-andrews.ac.uk/Biographies/Hardy.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

Mentoring Scheme, UK Mathematics Trust, School of Mathematics,
University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 mentoring@ukmt.org.uk www.ukmt.org.uk

Version 1.1, Apr 2021

1. On the Western Front in 1918, the guns fell silent at 11 a.m. on Monday 11 November. In 2018 the date of the centennial Remembrance Sunday was 11 November. In how many years in the period 1919 to 2018 inclusive did 11 November fall on a Sunday? You must provide a reasoned argument to justify your answer.
Note: Leap years have to be accounted for in the calculations. Every year whose year number is divisible by 4 is a leap year, with the exception of years ending in 00. For such a year to be a leap year, its number must be divisible by 400.

SOLUTION

Some preliminary remarks will be useful.

- Because $365 \equiv 1 \pmod 7$, in non-leap years the day of the week for a particular date moves on by one day.
- By the note following the question, 2000 was a leap year. Thus leap years occur at four-year intervals throughout the relevant period.
- It follows from the two preceding remarks that, every four years, the day of 11 November moves forward by five days.
- The day five days ahead is more easily located by going back two days.
- Since neither 1917 nor 1918 was a leap year, 11 November 1917 was a Sunday.
- Since 4 and 7 are coprime, the smallest positive integer divisible by both of them is $4 \times 7 = 28$. Hence the day-and-date pattern recurs every 28 years and the detailed working for the 28-year period 1917 to 1944 inclusive is shown in the table below.

2 Row	Four-year period	Day of last 11 November	One 11 November a Sunday?
1	1917 to 1920	Thursday	Yes
2	1921 to 1924	Tuesday	Yes
3	1925 to 1928	Sunday	Yes
4	1929 to 1932	Friday	No
5	1933 to 1936	Wednesday	Yes
6	1937 to 1940	Monday	No
7	1941 to 1944	Saturday	No

The following comments may clarify the information in the table.

a) You may want to check the information in Row 1. Note that 11 November ***does*** fall on a Sunday in this period, in 1917.

b) Row 1 tells us that 11 November was a Thursday in 1920. In Row 2 the successive 11 November days go forward in the pattern Friday, Saturday, Sunday, Tuesday. Sunday has been included, so we have Yes in the last column.

c) In Row 4 the successive 11 November days go forward in the pattern Monday, Tuesday, Wednesday, Friday. Sunday does not appear, so we have No in the last column.

d) In Row 6, Sunday has apparently been passed but checking the days shows that 11 November cannot be a Sunday in the years 1941 - 1944.

The table shows that 11 November fell on a Sunday 4 times between 1917 and 1944 inclusive. Since it was a Saturday in 1944, it follows, with sad appropriateness, that it was a Sunday in 1945. As far as the calculations are concerned, we see that the next 28-year period (1945 to 1972 inclusive) starts with a Sunday (compare 1917). This period will contribute 4 more Sundays, as will the next 28-year period (1973 to 2000 inclusive).

Finally the years 2001 to 2018 must be considered. The twelve years 2001 to 2012 inclusive correspond to the first three rows of the table and contribute 3 more Sundays. Row 4 shows that none of the years 2013 to 2016 contributes a Sunday. Further, that row also shows that 11 November was a Friday in 2016. Since neither 2017 nor 2018 is a leap year, 11 November was a Sunday in 2018, as expected. The Sundays enumerated contribute a total of $4 + 4 + 4 + 3 + 1 = 16$. The last step is to recall that the question asked for the number of relevant Sundays in the period 1919 to 2018 inclusive. This excludes 1917, so the final total is 15.

2. Prove that $n^5 - n$ is divisible by 30 for all integers n.

SOLUTION

This question is about integers and so factorisation, both of $n^5 - n$ and 30, is likely to be a useful tool. For simplicity, start by assuming that n is positive. The polynomial factorises as

$$n(n^4 - 1) = n(n^2 - 1)(n^2 + 1) = (n - 1)n(n + 1)(n^2 + 1)$$

and the prime factorisation of 30 is $30 = 2 \times 3 \times 5$.

The first three factors in the factorisation of the polynomial will always be three consecutive integers. It is quite acceptable simply to state that their product is divisible both by 2 and by 3. It therefore remains to consider divisibility by 5.

When divisibility needs to be considered, a useful tool is modular arithmetic. This because it is essentially the arithmetic of remainders. We also note that the factor $n^2 + 1$ has not yet been involved. If, for example, n has the form $5k + 2$ then

$$n^2 + 1 = (5k + 2)^2 + 1 = 25k^2 + 20k + 4 + 1 = 5(5k^2 + 4k + 1).$$

However, if n is a multiple of 5, then $n^2 + 1$ leaves remainder 1 on division by 5 and this shows that other factors will need to be considered. Working modulo 5 is not necessary but it provides a simple way of presenting the relevant results and is employed in the following table.

n	0	1	2	3	4
$n - 1$	4	0	1	2	3
$n + 1$	1	2	3	4	0
$n^2 + 1$	1	2	0	0	2

It only needs the observation that there is a zero in each column of the table to conclude that, whatever the value of $n^5 - n$, one of its factors is divisible by 5. Hence the same is true of $n^5 - n$ itself.

Finally recall that n was assumed to be a positive integer. The result is trivially true for $n = 0$. Again, if $n = -m$, where $m > 0$, then $n^5 - n = -(m^5 - m)$. This shows that the result is true for negative n and hence for all integers, as required.

3. In a certain country 10% of the workers get 90% of the total salary paid to the population. Supposing that the country is divided into a number of regions, is it possible that in every region the total salary of 10% of the workers is no greater than 11% of the total salary paid to the population of this region?

SOLUTION

The trick is to give the 10% of employees a very great deal of money and the 90% very little money. Since we are essentially seeking an example, it may possible to have a small number of regions, say two. The 90% figure also suggests that we make use of multiples of 9.

Suppose then, the country has two regions A and B. Suppose further that there are 10 employees in region A and 90 in region B. Let the annual salary of an employee in A be £81 000 and that of an employee in B be £1 000. The total amount earned by the employees in A is £810 000 and the total for B is £90 000. Thus the total amount of money paid out to all the employees is £900 000. Now 90% of this figure is £810 000 and this is the total amount paid out to the employees of A. Thus this scenario satisfies the condition in the first sentence of the question.

In A 10% of the employees is one person, earning £81 000. This is 10% of the total for the region, which is £810 000. We note that this is less than 11%.

In B 10% of the employees amounts to a total of nine people. These nine people together earn £9 000, which is 10% of the total for region B: it is also is less than 11%. An example satisfying all the conditions has been found.

4. *You may have met Mrs Logistic and her Breton cake plates in Question 6 of G H Hardy, Sheet 4. If so, you will not be surprised to hear that she is as keen on baking as she is on mathematics. This question involves the weighing scales that Mrs Logistic inherited from her mother-in-law.*
The scales have weights of 2 lb, 1 lb, 8 oz, 2 oz, 1 oz, ½ oz and ¼ oz. You will have noticed that the 4 oz weight is missing. Her favourite recipe is almond tart, which calls for 4¼ oz of ground almonds, 3½ oz of sugar, 2 eggs and the zest of 1 lemon for the filling. While Mrs L can estimate any fraction of the zest of a lemon, she cannot split an egg. Also for accuracy reasons she has to weigh each ingredient all together. Furthermore, she does not want to place any weights on the food tray. What is the minimum number of eggs she would require to make a multiple of this recipe using the scales that she has? (You may assume that her weights enable her to measure out the ingredients for the pastry case.)
Historical note: A pound (lb) contains 16 ounces (oz).

SOLUTION

• First notice that reducing the recipe is not possible, because the ground almonds require a measurement of ¼ oz. There is no fraction of this weight.

• Since the distribution of the weights is in the binary system, it would be a good idea to express the weights of the ingredients in binary form. We can also calculate their multiples in powers of 2 by shifting the binary point, as shown in the table below. For example, the 4.25 oz of ground almonds can be expressed in the binary system as

$$1 \times 2^2 + 0 \times 2^1 + 0 \times 2^0 + 0 \times 2^{-1} + 1 \times 2^{-2} \quad \text{written} \quad 100.01.$$

Note that the binary point marks the separation of non-negative powers of 2 from negative powers of 2. As an illustration the table shows what happens to the weights of ground almonds and sugar as the recipe is scaled up.

Recipe	Almond	Sugar
×1	100.01	11.1
×2	1000.10	111.0
×4	10001.00	1110.0
×8	100010.00	11100.0
×16	1000100.00	111000.0

• We want the smallest multiple of the basic recipe where neither of the measurements has a 1 in the third column to the left of the binary point.

• The basic recipe is rejected because of the ground almonds.

• We can't consider any multiple from 16 upwards, because this would require a weight of 4 lb.

• All powers of 2 up to 16 are rejected as well because of the sugar.

• All odd multiples up to 16 are rejected because of the ground almonds. (Make sure that you understand why this is so.) An alternative strategy would be to test all multiples up to a medium sort of value, say 10 or 12. The eventual solution shows why this happens to pay off.

• So we are left to test x6, x10, x12, x14.

• x6: ground almonds 11001.10, sugar 10101.0. The sugar requires 1 in the third column.

• x10: almond 101010.00, sugar 100011.0. Both have 0 in the third column.

• The minimum number of eggs is therefore 20.

5. *a*) Assuming that the quartic equation

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

has four real roots, show that the sum of the roots is equal to $-a$ and the product of the roots is equal to d. What are the corresponding expressions, in terms of the roots, for the coefficients b and c?

b) A student woke up at the end of an algebra class to hear the teacher say "... and I give you the hint that the roots are in arithmetic progression". Looking at the board, the student discovered a quartic equation to be solved for homework. Although he had time to copy only $x^4 + 14x^3 + 11x^2 - \cdots$ before the teacher erased the question, the sleepyhead was able to find all the roots. What are they?

Solution

a) Let the solutions of the equation be x_1, x_2, x_3 and x_4. Then

$$x^4 + ax^3 + bx^2 + cx + d \equiv (x - x_1)(x - x_2)(x - x_3)(x - x_4). \qquad (*)$$

When the product on the right-hand side of (*) is multiplied out, the term in x^3 is the sum of all the terms produced by taking x from three brackets and a $-x_i$ from the remaining bracket.

Comparing coefficients yields $a = -x_1 - x_2 - x_3 - x_4 = -(x_1 + x_2 + x_3 + x_4)$. Hence the sum of the roots is $-a$, as required. Similarly the constant term d is the product of the $-x_i$. Since there are four factors, this is non-negative and we see that $d = x_1x_2x_3x_4$.

To obtain the term in x^2, we take an x from two brackets in (*) and a pair of $-x_i$'s from the other two. Upon adding and comparing coefficients, we obtain

$$b = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4.$$

A similar argument involving the coefficient of x on the left-hand side shows that

$$c = -x_2x_3x_4 - x_1x_3x_4 - x_1x_2x_4 - x_1x_2x_3.$$

b) The teacher's remark implies that the roots can be denoted by $A - D$, A, $A + D$ and $A + 2D$. Using the first result in part *a*), we have the following equation.

$$A - D + A + A + D + A + 2D = -14. \tag{1}$$

Collecting like terms and cancelling in (1) yields

$$2A + D = -7 \qquad \text{or} \qquad D = -2A - 7. \tag{2}$$

We also know the coefficient of x^2, so

$$(A-D)A+(A-D)(A+D)+(A-D)(A+2D)+A(A+D)+A(A+2D)+(A+D)(A+2D) = 11. \tag{3}$$

Now (3) can be simplified to

$$A(2A + A + 2D) + A^2 - D^2 + (A + 2D) \times 2A = 11.$$

This in turn reduces to

$$A(5A + 6D) + A^2 - D^2 = 6A^2 + 6AD - D^2 = 11. \tag{4}$$

Now substitute from (2) into (4) to obtain $6A^2 - 6A(2A + 7) - (2A + 7)^2 = 11$. Rearranging this equation and dividing through by 10 produces the quadratic equation

$$A^2 + 7A + 6 = 0. \tag{5}$$

This has roots $A = -1$ and $A = -6$. Substituting $A = -1$ in (2) gives $D = -5$. These values produce the following as the roots of the incomplete quartic:

$$A - D = 4, \quad A = -1, \quad A + D = -6, \quad A + 2D = -11.$$

So what is the significance of the other values: $A = -6$ and $D = 5$? Do the substitution and find out what happens!

6. *a*) Prove that $\sqrt{3}$ is an irrational number.
Note: An irrational number is a number that is not rational, that is to say that it cannot be expressed as a fraction.

b) Prove that $2\sqrt{3}$ and $\sqrt{3} + \sqrt{2}$ are irrational.

c) Prove that $\sqrt[100]{\sqrt{3} + \sqrt{2}} + \sqrt[100]{\sqrt{3} - \sqrt{2}}$ is irrational.

Solution

a) There are many ways of proving results like this. The method presented below depends on the fact that each prime in the prime factorisation of a square appears to an even power. (See, for example, the note to G H Hardy, Sheet 6, Question 4.) We argue by contradiction and suppose that $\sqrt{3}$ is rational. Then it could be expressed in the form $\sqrt{3} = \frac{a}{b}$, where a and b are non-negative integers and $b \neq 0$. Therefore $3b^2 = a^2$. On the left-hand side the power of 3 appearing in the factorisation of b is even (and may be zero). Hence the power of 3 appearing in the factorisation of of the left-hand side is odd. However, the power of 3 appearing in the factorisation of the right-hand side is even. This is a ***contradiction*** and so we conclude that $\sqrt{3}$ is irrational.

b) If $2\sqrt{3}$ were rational, it could be expressed in the form $2\sqrt{3} = \frac{a}{b}$, where a and b are non-negative integers and $b \neq 0$. It would follow that $\sqrt{3} = \frac{a}{2b}$, which is rational. But this ***contradicts*** the result of part *a*) and we conclude that $2\sqrt{3}$ is irrational.

Again, assuming that $\sqrt{3} + \sqrt{2}$ is rational implies that $\sqrt{3} + \sqrt{2} = \frac{a}{b}$, where a and b are non-negative integers and $b \neq 0$. Squaring gives

$$3 + 2 + 2\sqrt{6} = \frac{a^2}{b^2} \qquad \text{or} \qquad \sqrt{6} = \frac{1}{2}\left(\frac{a^2}{b^2} - 5\right). \tag{*}$$

However an argument similar to that in part *a*) shows that $\sqrt{6}$ is irrational. (There is a slight complication because 3 is prime and 6 is not. Have a go at the details for yourself.) Thus we have another ***contradiction***, because the left-hand side of (*) is irrational and the right-hand side is rational. Hence $\sqrt{3} + \sqrt{2}$ is irrational.

c) To simplify the notation, put $a = \sqrt[100]{\sqrt{3} + \sqrt{2}}$ and $b = \sqrt[100]{\sqrt{3} - \sqrt{2}}$. Proof by contradiction has served us well so far, so assume $a + b$ is rational. Clearly

$$ab = \sqrt[100]{\left(\sqrt{3} + \sqrt{2}\right)\left(\sqrt{3} - \sqrt{2}\right)} = 1, \tag{**}$$

which is rational. The 100th roots are difficult to manipulate but, since $a^{100} = \sqrt{3} + \sqrt{2}$ and $b^{100} = \sqrt{3} - \sqrt{2}$, we see that $a^{100} + b^{100} = 2\sqrt{3}$. If it were possible to get from our assumption to $a^{100} + b^{100}$ is rational, then we would have the desired contradiction.

As a first step let us explore $a^n + b^n$ for smaller values of n.
First, by our assumption, $(a + b)^2$ is rational. But $(a + b)^2 = a^2 + b^2 + 2ab = a^2 + b^2 + 2$. It follows that $a^2 + b^2$ is rational.
Again, $(a + b)^3 = a^3 + b^3 + 3ab(a + b) = a^3 + b^3 + 3(a + b)$. Consequently $a^3 + b^3$ is rational. Here it is worth noting that we are aiming for $a^{100} + b^{100}$ and that 100 is even. Could we hop to $a^4 + b^4$ straight from $a^2 + b^2$?
We are assuming that $a + b$ is rational. On that assumption, $(a + b)^4$ is rational and we have shown that $a^2 + b^2$ is rational. Expand $(a + b)^4$; then rearrange and use (**). This yields

$$(a + b)^4 = a^4 + b^4 + 4(a^2 + b^2)ab + 6(ab)^2 = a^4 + b^4 + 4(a^2 + b^2) + 6.$$

Therefore $a^4 + b^4$ is rational and we have hopped to the next even power. Similar hops would take us to $a^{100} + b^{100}$. This is a an induction-style process. However, note that we are not pursuing a for-all-n argument but only going as far as $n = 100$. Nevertheless, it is worth considering a typical step from one even power to the next.

Suppose that we have shown that $a^n + b^n$ is rational for all even n satisfying $2 \leqslant n \leqslant 2k - 2$. Then, pairing terms as before, we can expand $(a + b)^{2k}$ as

$$a^{2k} + b^{2k} + \binom{2k}{1}(a^{2k-1}b + ab^{2k-1}) + \cdots + \binom{2k}{k-1}(a^{k+1}b^{k-1} + a^{k-1}b^{k+1}) + \binom{2k}{k}a^k b^k.$$

Here the pairing comes from the fact that $\binom{2k}{2k-r} = \binom{2k}{r}$. Rearranging this expansion gives

$$a^{2k} + b^{2k} + \binom{2k}{1}(a^{2k-2} + b^{2k-2})ab + \cdots + \binom{2k}{k-1}(a^2 + b^2)(ab)^{k-1} + \binom{2k}{k}(ab)^k.$$

Then we can use (**) again to obtain

$$(a + b)^{2k} = a^{2k} + b^{2k} + \binom{2k}{1}(a^{2k-2} + b^{2k-2}) + \cdots + \binom{2k}{k-1}(a^2 + b^2) + \binom{2k}{k}.$$

Since $(a + b)^{2k}$ and all the terms containing binomial coefficients have been shown to be rational, it follows that $a^{2k} + b^{2k}$ is rational. In particular $a^{100} + b^{100}$ is rational. However, it was earlier shown that $a^{100} + b^{100} = 2\sqrt{3}$ is irrational. This is the desired ***contradiction*** and hence we are justified in saying that

$$\sqrt[100]{\sqrt{3} + \sqrt{2}} + \sqrt[100]{\sqrt{3} - \sqrt{2}}$$

is irrational.

7. A function f is defined on the positive integers by $f(1) = 1$ and, for $n > 1$,

$$f(n) = f\left(\left\lfloor \frac{2n-1}{3} \right\rfloor\right) + f\left(\left\lfloor \frac{2n}{3} \right\rfloor\right),$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Is it true that $f(n) - f(n-1) \leqslant n$ for all integers $n > 1$?
[British Mathematical Olympiad Round 2 2012 Question 2]

Solution

In general, the dividing by 3 suggests that we consider the different possible remainders that can occur. First note that $f(2) = f(\lfloor \frac{3}{3} \rfloor) + f(\lfloor \frac{4}{3} \rfloor) = f(1) + f(1) = 2f(1) = 2$. Thereafter we consider three cases, where $k \geqslant 1$:

$$n = 3k + 2; \tag{1}$$

$$n = 3k + 1; \tag{2}$$

$$n = 3k. \tag{3}$$

From (1)

$$f(n) = f(3k + 2) = f\left(\left\lfloor \frac{6k + 4 - 1}{3} \right\rfloor\right) + f\left(\left\lfloor \frac{6k + 4}{3} \right\rfloor\right) = f(2k + 1) + f(2k + 1).$$

Thus
$$f(3k+2) = 2f(2k+1). \tag{4}$$
From (2)
$$f(n) = f(3k+1) = f\left(\left\lfloor \frac{6k+2-1}{3} \right\rfloor\right) + f\left(\left\lfloor \frac{6k+2}{3} \right\rfloor\right) = f(2k) + f(2k).$$
Thus
$$f(3k+1) = 2f(2k). \tag{5}$$
From (3)
$$f(n) = f(3k) = f\left(\left\lfloor \frac{6k-1}{3} \right\rfloor\right) + f\left(\left\lfloor \frac{6k}{3} \right\rfloor\right) = f(2k-1) + f(2k).$$
Thus
$$f(3k) = f(2k-1) + f(2k). \tag{6}$$
From (4) and (5) we now know that:
$$f(3k+2) - f(3k+1) = 2(f(2k+1) - f(2k)).$$
Similarly from (5) and (6)
$$f(3k+1) - f(3k) = f(2k) - f(2k-1).$$
To obtain $f(3k) - f(3k-1)$, note that $f(3k-1) = f(3(k-1)+2)$. Then (4) shows that this is equal to
$$f(3(k-1)+2) = 2f(2(k-1)+1) = 2f(2k-1)).$$
Hence
$$f(3k) - f(3k-1) = f(2k-1) + f(2k) - 2f(2k-1) = f(2k) - f(2k-1).$$
It is now a matter of applying these rules.
$$f(2) - f(1) = 1$$
$$f(3) - f(2) = f(2) - f(1) = 2 - 1 = 1$$
$$f(4) - f(3) = f(2) - f(1) = 2 - 1 = 1$$
$$f(5) - f(4) = 2(f(3) - f(2)) = 2 \times 1 = 2$$
Note that at this last stage there is a doubling of the previous value and so it is likely that the difference becomes big enough when it is of the form $f(3k+2) - f(3k+1)$. Some experimenting with intermediate values (try for yourself) shows that
$$f(242) - f(241) = f(3 \times 80 + 2) - f(3 \times 80 + 1) = 2(f(2 \times 80 + 1) - f(2 \times 80)) = 256.$$
Since $256 > 242$, we see that that it is not always true that $f(n) - f(n-1) \leqslant n$.

8. *a*) Find integers x and y that satisfy the equation $18x + 5y = 48$.

b) Is there a pair of positive integers x and y satisfying the equation $18x + 5y = 48$? Is this pair the only such solution?

c) If a cock is worth 5 coins, a hen 3 coins and 3 chicks together 1 coin, then how many cocks, hens and chicks, totalling 100, can be bought for 100 coins?
This problem appears in the Mathematical Classic of Zhang QiuJian, who flourished in China in the second half of the fifth century CE.

Solution

You may find parts of this solution unnecessarily laborious. If so, then ensure you read the remark at the end.

a) The form of the equation is like expressions that occurred in Question 8 of G H Hardy, Sheet 6 and this suggests that we turn to the Euclidean algorithm, applied to 18 and 5. The divisions are as follows:

(1) $18 = 5 \times 3 + 3$

(2) $5 = 3 \times 1 + 2$

(3) $3 = 1 \times 2 + 1$

Tracking back through the remainders:

$$1 = 3 - 2 = 3 - (5 - 3) = 2 \times 3 - 5 = 2(18 - 5 \times 3) - 5 = 2 \times 18 - 7 \times 5.$$

This can be rewritten as $18 \times 2 + 5 \times (-7) = 1$. Multiplying through by 48 yields

$$18 \times 2 \times 48 + 5 \times (-7) \times 48 = 18 \times 96 + 5 \times (-336) = 48.$$

Hence one solution in integers to the equation $18x + 5y = 48$ is $x = 96$ and $y = -336$.

b) In this part we have the problem of constructing positive integers for the solution. There is a hint in the discussion in Question 8 of G H Hardy, Sheet 6. On the left-hand side of the relation $18 \times 96 + 5 \times (-336) = 48$, we could consider adding and subtracting the product 18×5. However this is much smaller than -336. To gauge how large a term we need, we could try a multiple, say $18 \times 5t$.

Our relation now becomes

$$18 \times 96 - 18 \times 5t + 18 \times 5t + 5 \times (-336) = 48.$$

This can be rewritten as

$$18(96 - 5t) + 5(18t - 336) = 48 \qquad (*)$$

To obtain positive solutions for the part *b*) equation, we require the inequalities $96 - 5t > 0$ and $18t - 336 > 0$ to be satisfied simultaneously. The first yields $t < 19\frac{1}{5}$. The second is equivalent to $3t - 56 > 0$ or $t > 18\frac{2}{3}$. The only integer that satisfies both these conditions is $t = 19$. Substituting this value into the brackets in (*) gives $96 - 5t = 96 - 95 = 1$ and $18t - 336 = 342 - 336 = 6$. Then (*) becomes $18 \times 1 + 5 \times 6 = 48$. This shows that a solution in positive integers for the original equation is $x = 1$ and $y = 6$.

We might well have spotted this solution but it would then be a lot harder to show that there is just one solution of this type. The argument above provided just one value for t and so $x = 1$ and $y = 6$ is the only solution of the form $x = 96 - 5t$ and $y = 18t - 336$. This does not rule out the possibility that there are other solutions of a different form. However, it turns out that there are none and so $x = 1$ and $y = 6$ is the only solution in positive integers. You may want to pursue this aspect further or discuss it with your mentor.

c) Suppose that x denotes the number of cocks, y the number of hens and z the number of chicks. The situation is described by the equations

$$5x + 3y + \frac{1}{3}z = 100 \qquad \text{and} \qquad x + y + z = 100.$$

Eliminating one of the variables yields $5x + 3y + \frac{1}{3}(100 - x - y) = 100$. On multiplying up and simplifying we have $14x + 8y = 200$ or

$$7x + 4y = 100. \tag{1}$$

Using the method of part *a*) on (1) gives the solution $x = -100$, $y = 200$. Extending the working as in part *b*) shows that the following expressions also give solutions for x and y:

$$x = 4t - 100 \qquad \text{and} \qquad y = 200 - 7t. \tag{2}$$

The corresponding value for z is

$$z = 100 - (4t - 100) - (200 - 7t) = 3t. \tag{3}$$

We require the numbers of cocks, hens and chicks to be positive and by (2) and (3) this entails

$$t > 25 \qquad \text{and} \qquad t < 28\frac{4}{7} \qquad \text{and} \qquad 3t > 0. \tag{4}$$

To satisfy these equations requires that t take the values 26, 27 or 28. Using $t = 26$ gives the solutions

$$x = 4 \times 26 - 100 = 4 \quad \text{and} \quad y = 200 - 7 \times 26 = 200 - 182 = 18 \quad \text{and} \quad z = 3 \times 26 = 78.$$

You may want to see what happens when $t = 27$ and $t = 28$. (Note that Zhang gave these three sets of values in his solution.) And finally, there there remains the question of whether these are the only solutions in positive integers.

Remark: In part a) you may well have spotted the result $18 \times 2 + 5 \times (-7) = 1$ *without the use of the algorithm. Again, in b) you may have spotted the solution in positive integers. Note that it is the only solution of this type because* $x = 2$ *gives a non-integer value for* y *and* $x > 2$ *forces* $y < 0$. *However, the extended working is given as it provides a method for the more elaborate problem in part c).*

United Kingdom
Mathematics Trust

Mentoring Scheme

Supported by OxFORD ASSET MANAGEMENT

G. H. Hardy

Sheet 8

Solutions and comments

This programme of the Mentoring Scheme is named after G. H. Hardy (1877–1947).

See http://www-history.mcs.st-andrews.ac.uk/Biographies/Hardy.html for more information.

Enquiries about the Mentoring Scheme should be sent to:

Mentoring Scheme, UK Mathematics Trust, School of Mathematics,
University of Leeds, Leeds LS2 9JT

☎ 0113 343 2339 mentoring@ukmt.org.uk www.ukmt.org.uk

Version 1.0, Dec 2018

1. Question 8 on both G H Hardy Sheet 6 and G H Hardy Sheet 7 involved the use of the ***Euclidean Algorithm.*** Now suppose a friend asks you about it. Think it through and write down your explanation. Make clear what the algorithm does and how your friend should operate it.
The following examples might help you to crystallise the ideas. You could indicate whether each of the following is possible. If so, please write down the values of m and n; if not, please explain why no such m and n exist.

a) $3247m + 3927n = 1$

b) $3247m + 4389n = 1$

c) $3521m + 2863n = 7$

d) $4380m + 3015n = 5$

e) $4380m + 3015n = 30$

Solution

There are all sorts of explanations that you could give to your friend but any such should contain the following points.

- Make sure that your friend can run the algorithm. For a simple example, see Question 8*a)* of G H Hardy, Sheet 6.
- The Euclidean Algorithm will find the gcd (greatest common divisor) of two integers. The gcd appears as the last non-zero remainder in the divisions in the first part of the algorithm.
- The examples given throw light on the process in the following ways.
  In *a)*, the two given integers have gcd equal to 17. There are no integers m and n satisfying the given equation: notice that the left-hand side of the given relation is divisible by 17 but 1 is not.
  In *b)* the two integers are coprime (their gcd is 1) and the Euclidean Algorithm yields the relation $4389 \times 907 + 3247 \times (-1226) = 1$.
  In *c)* both integers are clearly divisible by 7. Running the algorithm confirms that this is the gcd and that $3521 \times 161 + 2863 \times (-198) = 7$.
  In *d)* the integers are both divisible by 3 and by 5. Thus the left-hand side of the given relation is divisible by 15 but note that 5 on the right is not. There are no integers m and n satisfying the given equation.
  In *e)* the Euclidean Algorithm will show that 15 is indeed the gcd. Hence there will exist integers m and n with $4380m + 3015n = 15$. Doubling these values will produce the relation in *e)*.
- We have not yet seen anything that amounts to a ***proof*** that the last non-zero remainder produced by the algorithm is in fact the gcd of the two initial integers. You might like to look up a proof or discuss this point with your mentor or, indeed, devise one of your own.

2. At a certain American University, the Mathematics Faculty consists of the departments of Mathematics, Statistics and Computer Science. There are two male and two female professors in each department and the Faculty Committee must contain six professors. Professor G-N, the Faculty Chair, wants to canvass as wide a range of opinions as possible. He therefore requires that the Committee contain three men and three women and also two professors from each of the three departments. Find the number of possible committees that can be formed subject to these requirements.

Solution

If exactly one female professor is chosen from each department, then one male professor from each department must also be chosen. There are $2^6 = 64$ ways to form the committee in this manner.

Alternatively, suppose that two female professors are chosen from the same department. Then two male professors have to be chosen from one of the other two departments. (Remember that we have to fit in another female professor.) The committee will then be completed by choosing one female and one male professor from the remaining department. There are $3 \times 2 \times 2 \times 2 = 24$ ways to form the committee in this manner. Overall the number of committees that meet the requirements is $64 + 24 = 88$.

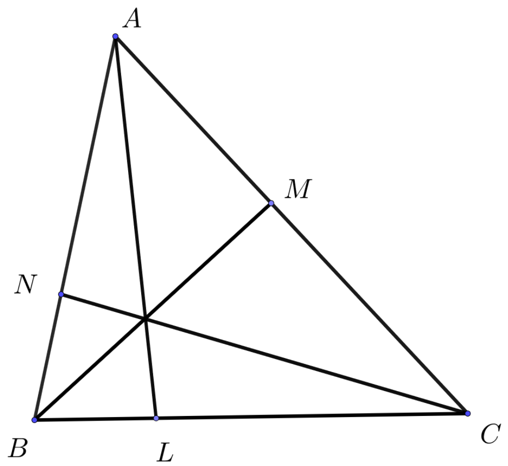
3. Let L, M and N be points on the sides BC, CA and AB (respectively) of the triangle ABC. Prove that the lines AL, BM and CM are concurrent if and only if

$$\frac{AN}{NB} \times \frac{BL}{LC} \times \frac{CM}{MA} = 1. \qquad (*)$$

Deduce that the altitudes of a triangle are concurrent.
Note: For the definition of ***altitude****, you can consult G H Hardy, Sheet 5, Question 3.*

Solution

The “if and only if" appearing in the question indicates that a two-part proof will be needed. So let us start by assuming that AL, BM and CN are concurrent (pass through a point). The configuration is as shown below. Denote the point common to all three lines by P.



The triangles BPL and CPL have a common height and so their areas are proportional to their

bases. This can be written as

$$\frac{BL}{LC} = \frac{[BPL]}{[CPL]}. \tag{1}$$

Here the square brackets indicate area so $[BPL]$ is the area of the triangle BPL. This is a standard notation for the area of a polygon. However, (1) is not very useful as it stands: if we carry out the same procedure for the other two sides of triangle ABC, then we have six disjoint triangles and no way of linking their areas. But there are two other triangles with bases BL and LC, namely BAL and CAL. They also have a common height and so we can write

$$\frac{BL}{LC} = \frac{[BAL]}{[CAL]}. \tag{2}$$

Combining (1) and (2) yields

$$\frac{BL}{LC} = \frac{[BPL]}{[CPL]} = \frac{[BAL]}{[CAL]} = \frac{[BAL]-[BPL]}{[CAL]-[CPL]} = \frac{[BPA]}{[CPA]}. \tag{3}$$

A word of caution is appropriate here: the penultimate step in (3) does not look like the usual subtraction of fractions. However, it is valid because the ratios involved are the same. You should check the validity of this for yourself.

By applying the same procedure to the other two sides of triangle ABC, we obtain

$$\frac{CM}{MA} = \frac{[CPB]}{[APB]} \tag{4}$$

and

$$\frac{AN}{NB} = \frac{[APC]}{[BPC]}. \tag{5}$$

We can now combine (3), (4) and (5) to obtain

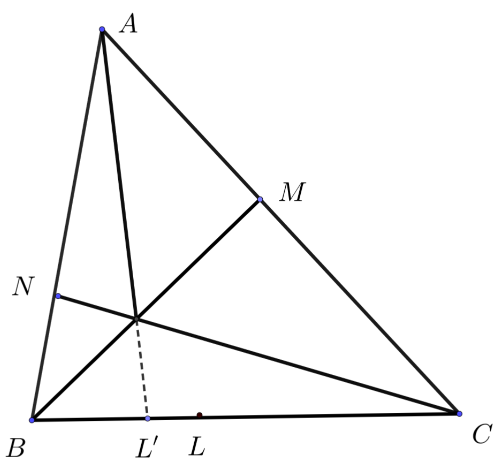
$$\frac{BL}{LC} \times \frac{CM}{MA} \times \frac{AN}{NB} = \frac{[BPA]}{[CPA]} \times \frac{[CPB]}{[APB]} \times \frac{[APC]}{[BPC]}. \tag{6}$$

The order of the vertices in the ratios on the right hand side is immaterial, since

$$[BPA] = [APB] = \text{area of triangle } ABP.$$

We now see that the product on the right-hand side of (6) is 1 and so the result in (*) is proved.

Now assume that (*) is satisfied; we have to prove that AL, BM and CN are concurrent. The relevant diagram is shown below.



In this diagram BM and CN are shown; again denote their point of intersection by P. The points A and P are joined and the resulting segment is produced (extended) to meet BC in L'. We shall show that L and L' are the same point.

We note that AL', BM and CN are concurrent so, by the preceding result,

$$\frac{BL'}{L'C} \times \frac{CM}{MA} \times \frac{AN}{NB} = 1.$$

But we are also assuming that (*) is satisfied and consequently

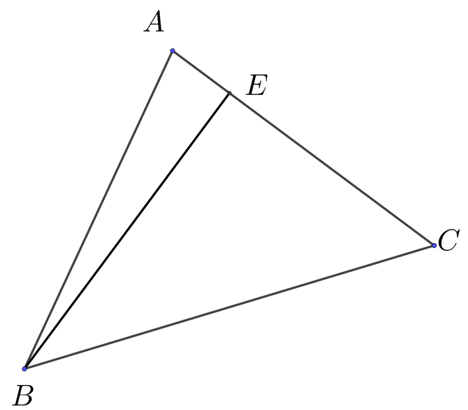
$$\frac{BL'}{L'C} \times \frac{CM}{MA} \times \frac{AN}{NB} = \frac{BL}{LC} \times \frac{CM}{MA} \times \frac{AN}{NB},$$

since both products are equal to 1. Cancelling yields

$$\frac{BL'}{L'C} = \frac{BL}{LC}.$$

It follows that L and L' must be the same point and so the concurrency is proved.
But see Note 2 below.

The altitudes of triangle ABC are lines from each vertex perpendicular to the opposite side. In the diagram below, just the altitude BE from B is shown, with $\measuredangle BEC$ being a right angle.



To simplify the notation, let us use the standard notation for the sides of the triangle (where $BC = a$ and so on). Then basic trigonometry shows that

$$\frac{CE}{EA} = \frac{a\cos C}{c\cos A}. \tag{7}$$

The foot of the altitude through A is traditionally denoted by D and the foot of that through C by F. We now have two comparable ratios, namely

$$\frac{BD}{DC} = \frac{c\cos B}{b\cos C} \quad \text{and} \quad \frac{AF}{FB} = \frac{b\cos A}{a\cos B}. \tag{8}$$

(You may want to draw a complete diagram for yourself to check that the ratios in (8) are correct.) Multiplying all three ratios from (7) and (8) together produces

$$\frac{BD}{DC} \times \frac{CE}{EA} \times \frac{AF}{FB} = \frac{abc\cos A\cos B\cos C}{abc\cos A\cos B\cos C} = 1.$$

It follows from the second result proved above that the altitudes AD, BE and CF are concurrent.
Note 1: The first result proved is called Ceva's theorem; the second result above is its converse.
Note 2: In the cases considered here the points such as L, M and N are internal to the sides of the triangles and the points of concurrency are inside the triangles themselves. You may want to experiment with other possible configurations to see if the results still hold. Something to discuss with your mentor perhaps.
Note 3: Look back to the altitude proof in Sheet 5. Take a moment to check how special it was, depending on the existence of so many right angles in the configuration. Nevertheless, this question shows that it can be considered as a special case of a more general result.

4. Let S be the increasing sequence of positive integers whose binary representation has exactly eight ones. Let N be the 1000th number in S. Find the the integer N in decimal form.

SOLUTION

In this question we have to find out where the 1000th number N comes in the sequence S of positive integers expressed in binary form. The initial investigation can be conveniently presented in the table below and some explanatory comments on the entries follow the table.

Range	N° of binary digits	N° of expressions with eight 1's	Running total
2^7 to $2^8 - 1$	8	$\binom{8}{0} = 1$	1
2^8 to $2^9 - 1$	9	$\binom{8}{1} = 8$	9
2^9 to $2^{10} - 1$	10	$\binom{9}{2} = \frac{9\times 8}{2\times 1} = 36$	45
2^{10} to $2^{11} - 1$	11	$\binom{10}{3} = \frac{10\times 9\times 8}{3\times 2\times 1} = \frac{10}{3}\binom{9}{2} = 120$	165
2^{11} to $2^{12} - 1$	12	$\binom{11}{4} = \frac{11}{4}\binom{10}{3} = 330$	495
2^{12} to $2^{13} - 1$	13	$\binom{12}{5} = \frac{12}{5}\binom{11}{4} = 792$	1287

- Note that all numbers in the table are in decimal form.
- Next, we cannot have eight 1's until our numbers have at least eight binary digits. This is true of integers in the range 2^7 to $2^8 - 1$ and this range appears in the first row of the table. The only relevant number is (in binary) 11111111 and recall that $\binom{n}{0}$ is defined to be 1 for all positive integers n.
- It is very important to realise that the binary expression of any number in the table ***must*** start with a 1. For example, in the penultimate row all the 495 numbers have 12 binary digits and the first digit is 1. This initial digit is followed by eleven others that may be either 1 or 0.
- The calculations may be explained with reference to the third row, where the numbers have 10 binary digits and all start with 1. The remaining 9 binary digits consist of 7 1's together with 2 0's. These may be arranged in $\binom{9}{2}$ ways. All the subsequent binomial coefficients depend on the same argument. In general, the number of arrangements of $m + n$ objects,

where m are of one type and n of another is

$$\binom{m+n}{m} = \binom{m+n}{n} = \frac{(m+n)!}{m!n!}.$$

[Make sure that you understand why this is so.]

- Notice how the calculations in the last three rows have been shortened by hopping from one to the next.

Though the table is a way of presenting the calculations in a succinct form, it was still laborious to construct. It shows that there are 495 numbers in the sequence S in the range 2^7 to $2^{12} - 1$. But it so happens that $495 = \binom{12}{8}$ and this must be more than a coincidence. It can be explained by the observation that $\binom{12}{8}$ is the number of strings of 12 digits consisting of 8 1's and 4 0's, where we do ***not*** have the restriction that the string should start with 1. This covers all the numbers in the first five rows of the table. Thus all the displayed calculations can be reduced to just one. To sum up: we have 495 numbers in S that are smaller than 2^{12} and so we have 505 more numbers to find.

Turning to the last row of the table, we note that the running total passes 1000. Thus the number we seek has 13 digits and starts with a 1 but we do not need all numbers of this form. What about numbers starting 10? Another table could be a good way of recording further calculations.

Starting	Nº of expressions with eight 1's	How many to find?
10	$\binom{11}{4} = \frac{11\times10\times9\times8}{4\times3\times2\times1} = 330$	$505 - 330 = 175$
11	$\binom{11}{5} = 462$	Not all needed
110	$\binom{10}{4} = 210$	Not all needed
1100	$\binom{9}{3} = 84$	$175 - 84 = 91$
1101	$\binom{9}{4} = 126$	Not all needed
11010	$\binom{8}{3} = 56$	$91 - 56 = 35$
11011	$\binom{8}{4} = 70$	Not all needed
110110	$\binom{7}{3} = 35$	$35 - 35 = 0$

We conclude that the 1000th number is the largest number in S which starts 110110. This number is 1101101111000 and it remains to convert this binary string to decimal form. It is

$$2^{12} + 2^{11} + 2^9 + 2^8 + 2^6 + 2^5 + 2^4 + 2^3 = 4096 + 2048 + 512 + 256 + 64 + 32 + 16 + 8 = 7032.$$

5. Let x and y be real numbers such that $\dfrac{\sin x}{\sin y} = 3$ and $\dfrac{\cos x}{\cos y} = \dfrac{1}{2}$. Find the value of $\dfrac{\sin 2x}{\sin 2y} + \dfrac{\cos 2x}{\cos 2y}$.

Solution

The first term in the required sum is comparatively easy to find. We have

$$\frac{\sin 2x}{\sin 2y} = \frac{2\sin x\cos x}{2\sin y\cos y} = \frac{\sin x}{\sin y} \times \frac{\cos x}{\cos y} = \frac{3}{2}. \tag{1}$$

The second term is more complicated because the double angle formula for cos involves a difference and a square. It can sometimes be easier to navigate a complex algebraic manipulation if the terms are denoted by algebraic symbols. With this in mind, set $\dfrac{\sin x}{\sin y} = A$ and $\dfrac{\cos x}{\cos y} = B$. Then(1) becomes

$$\frac{\sin 2x}{\sin 2y} = AB. \tag{2}$$

Again,

$$\frac{\cos 2x}{\cos 2y} = \frac{2\cos^2 x - 1}{2\cos^2 y - 1}. \tag{3}$$

The presence of the squares suggests that we go to the most basic of all trigonometrical relations. By the definitions of A and B, we have

$$1 = \sin^2 x + \cos^2 x = A^2\sin^2 y + B^2\cos^2 y = A^2(1-\cos^2 y) + B^2\cos^2 y. \tag{4}$$

From this we can obtain an expression for $\cos^2 y$ and it is

$$\cos^2 y = \frac{A^2-1}{A^2-B^2}. \tag{5}$$

It follows that, since $\cos x = B\cos y$,

$$\cos^2 x = \frac{B^2(A^2-1)}{A^2-B^2}. \tag{6}$$

Equations (5) and (6) now yield

$$2\cos^2 y - 1 = \frac{2(A^2-1)}{A^2-B^2} - 1 = \frac{2(A^2-1)-(A^2-B^2)}{A^2-B^2}. \tag{7}$$

and

$$2\cos^2 x - 1 = \frac{2B^2(A^2-1)}{A^2-B^2} - 1 = \frac{2B^2(A^2-1)-(A^2-B^2)}{A^2-B^2}. \tag{8}$$

Now substitute from (7) and (8) into (3) to obtain, after cancelling the denominators A^2-B^2,

$$\frac{\cos 2x}{\cos 2y} = \frac{2B^2(A^2-1)-(A^2-B^2)}{2(A^2-1)-(A^2-B^2)} = \frac{2A^2B^2-B^2-A^2}{A^2+B^2-2}. \tag{9}$$

Incorporating (2), we have

$$\frac{\sin 2x}{\sin 2y} + \frac{\cos 2x}{\cos 2y} = AB + \frac{2A^2B^2-B^2-A^2}{A^2+B^2-2}. \tag{10}$$

Now all that remains is to set $A = 3$ and $B = \frac{1}{2}$ in (10). When this done and the calculations are completed, we have the final result

$$\frac{\sin 2x}{\sin 2y} + \frac{\cos 2x}{\cos 2y} = \frac{49}{58}.$$

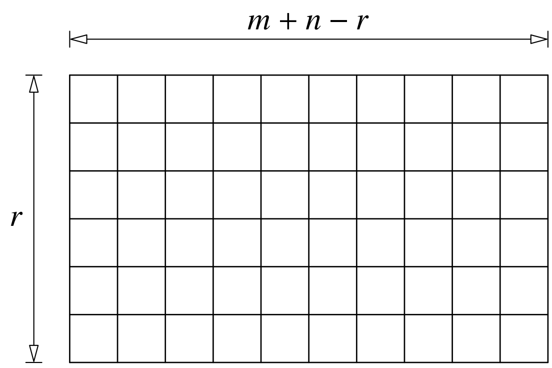
6. Jane is a recent maths graduate who who has obtained a parliamentary internship. reporting to Ms X, MP. She has asked Jane to work on a piece of legislation, which is coming up for debate. Jane has lined up m supportive arguments and n which are against the proposed measure. She advises Ms X that, if she presents r arguments, then there are $\binom{m+n}{r}$ many different combinations of pros and cons. (Here she reminds Ms X that this requires $r < m + n$.) Moreover there are $\binom{m}{k}\binom{n}{r-k}$ different combinations when k many pro and $r - k$ many con arguments are included. Jane shows Ms X the following relation

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} \qquad (*)$$

but Ms X looks increasingly nervous. Please reassure Ms X by explaining the situation to her.

Solution

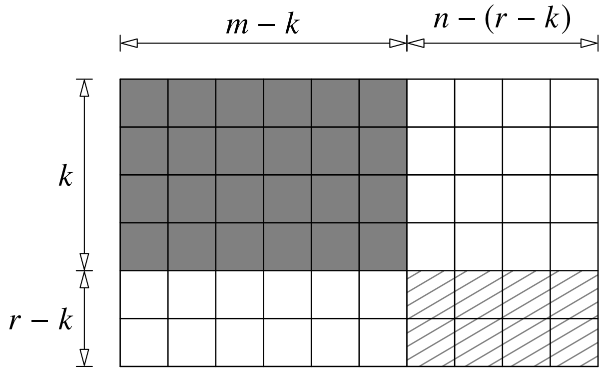
Two different arguments will be presented, one geometric and the other algebraic.
Geometric: Number all the arguments from 1 to $m + n$. If we were going to go through each argument and select r of them, then we know there are $m + n$ many steps in this task. If we were to pick r of the arguments and reserve $m + n - r$ arguments, we could represent this in a rectangular grid of dimensions $r \times (m + n - r)$.



Assume this grid has r rows and $m + n - r$ columns, as shown in the diagram. In going from the upper left corner to the bottom right corner there are $m + n$ steps to take, steps along the grid, that is. The numbered arguments are one to one correspondence to the $m + n$ steps taken. Whenever we choose an argument, for the corresponding move we go down. Whenever we pass by an argument and refrain from choosing it, we move to the right. This gives the left hand side of the equality.

Without loss of generality, we can number all m pro arguments as 1 to m and the n con arguments as $m + 1$ to $m + n$. If we choose $k < r$ pro arguments and $r - k$ con arguments, then the grid become subdivided into two independent parts, as shown in the diagram below. To go from top left to bottom right, we take $m - k$ steps to the right and k steps down. This involves the darker

rectangle and we end up at the joint vertex of the two shaded rectangles. Similarly, we go on from this point to the lower right corner.



The division of the grid into two subgrids is entirely determined by the value of k where $0 < k < r$. When we sum over all possible values of k we obtain the right hand side of the relation (*).

Algebraic:
When we carry out a bracket expansion we consider one term from each bracket at a time. In other words, the expanded expression before simplifying is the sum of terms, each of which has exactly one term from each bracket. Now instead of thinking of terms in a bracket as numbers, we can think of them as being “choose" and “not choose". That is to say, we either choose the first term in an individual bracket - or we do not choose it and choose the second term instead. To make life easy we can use 1 as “not choose" and x as “choose". If you find this confusing, see the note after the solution.

So now we can treat each argument like a bracket and there are $m+n$ brackets containing $(1+x)$. Expanding by the binomial theorem, we know this is

$$(1+x)^{m+n} = \sum_{r=0}^{m+n} \binom{m+n}{r} x^r. \tag{1}$$

This can be re-written as

$$(1+x)^m(1+x)^n = \sum_{i=0}^{m} \binom{m}{i} x^i \ \sum_{j=0}^{n} \binom{n}{j} x^j \tag{2}$$

Now we are going to merge the two summations in (2). In the back of our minds we think of some large bracket expansion. First we note that the powers of x are going to range from 0 to $m+n$. Also, for each value of r there are “k" many brackets from $(1+x)^m$ and “$r-k$" many brackets from $(1+x)^n$, where k ranges from 0 to r. It follows that

$$\sum_{i=0}^{m} \binom{m}{i} x^i \ \sum_{j=0}^{n} \binom{n}{j} x^j = \sum_{r=0}^{m+n} \left(\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} \right) x^r \tag{3}$$

If we compare the inner summation, (what is within the bracket on the right-hand side of (3)) with the coefficient term on the right-hand side of (1), we have the relationship we wish to prove.

Note: If you haven’t studied the use of the binomial expansion to analyse the probabilities of coin tossing, you could look at https://www3.nd.edu/ rwilliam/stats1/x13.pdf This considers p and q as two numbers adding up to 1 and calculates probabilities involving numbers of heads

and tails. What we are doing here is similar, except there are no probabilities involved, we are only counting different ways of choosing. Therefore p and q are just symbols and, in order to facilitate our calculations, we use 1 and x because we know the indices involved add up to $m + n$. If one of them has a power to r, then the other one has a power to $m + n - r$ and there is no need to carry both terms around. So now we can treat each argument as a bracket and there are $m + n$ brackets of $(1 + x)$.

7. Naomi and Tom play a game, with Naomi going first. They take it in turns to pick an integer from 1 to 100, each time selecting an integer which no-one has chosen before. A player loses the game if, after their turn, the sum of all the integers chosen since the start of the game (by both of them) cannot be written as the difference of two square numbers. Determine if one of the players has a winning strategy, and if so, which.
[British Mathematical Olympiad Round 1 2016 Question 4]

Solution

This problem falls into two parts: identifying those integers which cannot be written as a difference of two squares and then finding a winning strategy. The first part was discussed in Hanna Neumann, Sheet 2, Question 2. For those who did not come across that question, a short justification follows.

An integer which is a difference of two squares can be written as $n^2 - m^2 = (n - m)(n + m)$. If n and m are both odd or both even, then their sum and difference are both even and so $n^2 - m^2$ is divisible by 4. If n and m are of different parity, then their sum and difference are both odd and so $n^2 - m^2$ is odd. It looks as though differences of two squares are either divisible by 4 or they are odd. However, to confirm this, we must check the reverse implication.

If N is a multiple of 4, then $N = 4k$ and we can write $N = (k + 1)^2 - (k - 1)^2$. Alternatively, if N is odd, say $N = 2k + 1$, then we have $N = (k + 1)^2 - k^2$. We now know that differences of two squares are either multiples of 4 or they are odd. A more succinct statement is that differences of two squares are precisely those N such that $N \equiv 2 \pmod 4$. Thus a player will lose if they are forced to choose an integer that is congruent to 2 modulo 4.

It is possible that making the the first move will put Naomi in the stronger position. She can start by selecting 100. On each of Naomi's subsequent turns, she can select $100 - T$, where T is the number Tom chose on the previous turn. Naomi will not lose as the total will be divisible by 4 after each of her turns. She will always be able to play, choosing a new integer each time, because, if Tom chooses 50, then he will lose. Thus Tom will eventually be the first player to choose an integer congruent to 2 modulo 4 and this will make make the overall total congruent to 2 modulo 4.

8. Several positive integers are written in a row. In each move, Bob chooses any two adjacent numbers in which the one on the left is greater than the one on the right, doubles each of them and then switches them around. Prove that Bob can make only a finite number of such moves.

Solution

Give each number in the row a label $a, b, c, \ldots$ etc that sticks with it as it moves in the list and changes value (due to the doubling). At each step of the process we exchange the numbers

corresponding to two labels. Let $n_t(x)$ be the number associated with label x after step t. A very helpful preliminary to a solution is to try out some lists of integers and to see what happens. You may begin to suspect that any pair of labels cannot be swapped more than once. This turns out to be true but, so far, is rather roughly phrased. We replace the statement by the following lemma. (A ***lemma*** is a supplementary result.)

Lemma For each pair of labels, $\{x, y\}$, a number associated with x and an a number associated with y are exchanged at most once.

We prove the lemma by contradiction. Suppose it is not true. Let $\{x, y\}$ be the first pair of labels that are exchanged twice. Denote the steps for these first two exchanges by t_1 and t_2. Consider the state directly after step t_1. Without loss of generality, the sequence of labels at this point is $\ldots x, y \ldots$ and the corresponding numbers are $\ldots n_{t_1}(x), n_{t_1}(y) \ldots$ where $n_{t_1}(x) < n_{t_1}(y)$.

Between steps t_1 and t_2 we do not exchange labels x and y and so x stays to the left of y. Hence right before step t_2 the sequence of labels is $\ldots x, y \ldots$ and $n_{t_2-1}(x) > n_{t_2-1}(y)$. Hence label x must have doubled more than y during steps $t_1 + 1, \ldots, t_2 - 1$. Moreover, by assumption each pair of labels is exchanged at most once during these steps. Hence there must be a label z that exchanges with x during these steps but which does not exchange with y.

Since x and y are next to each other after step t_1 *and* after step $t_2 - 1$ the sequence of labels after both these steps must have the form $\ldots, z, \ldots, x, y, \ldots$. The first exchange of z and x puts z to the right of x. Hence there must be a *second* exchange of x and z before step t_2 that puts z back to the left of x. This contradicts our assumption that x and y are the first pair of labels to be exchanged twice.

Summing up, assuming the lemma result to be false has led to a contradiction. Thus the lemma result must be true and any two labels can be swapped at most once. Thus Bob will run out of possible swaps after a finite number of moves and the required result is proved.